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The Plancherel measure for p -forms in real hyperbolic spaces

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Abstract

The Plancherel measure is calculated for antisymmetric tensor fields (p -forms) on the real hyperbolic space H^N . The Plancherel measure gives the spectral distribution of the eigenvalues ω_λ of the Hodge-de Rham operator $\Delta = d\delta + \delta d$. The spectrum of Δ is purely continuous except for N even and $p = \frac{1}{2}N$. For N odd the Plancherel measure $\mu(\lambda)$ is a polynomial in λ^2 . For N even the continuous part $\mu(\lambda)$ of the Plancherel measure is a meromorphic function in the complex λ -plane with simple poles on the imaginary axis. A simple relation between the residues of $\mu(\lambda)$ at these poles and the (known) degeneracies of Δ on the N -sphere is obtained. A similar relation between $\mu(\lambda)$ at discrete imaginary values of λ and the degeneracies of Δ on S^N is found for N odd. The p -form ζ -function, defined as a Mellin transform of the trace of the heat kernel, is considered. A relation between the ζ -functions on S^N and H^N is obtained by means of complex contours. We construct square-integrable harmonic k -forms on H^{2k} . These k -forms contribute a discrete part to the spectrum of Δ and are related to the discrete series of $SO_0(2k, 1)$. We also give a group-theoretic derivation of $\mu(\lambda)$ based on the Plancherel formula for the Lorentz group $SO_0(N, 1)$.

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1. Introduction

In quantum field theory one often needs to give a meaning to formal expressions like determinants of second-order differential operators. A convenient regularization procedure is the ζ -function method [12,17]. This method can be applied when the spacetime in which the quantum field propagates admits a Euclidean section, i.e., when a Wick rotation of the time variable from the real axis to the imaginary axis defines a Riemannian manifold M with positive definite metric [34]. The one-loop functional determinant of an operator like $-\nabla^\alpha \nabla_\alpha + \kappa$ can then be defined from the ζ -function of the corresponding elliptic operator on M according to [17]

$$\det[-\nabla^\alpha \nabla_\alpha + \kappa] = \exp[-\zeta'(0)]. \quad (1.1)$$

In case the Euclidean section is compact the ζ -function is given by

$$\zeta(z) = \sum_{n=1}^{\infty} d_n \omega_n^{-z}, \quad (1.2)$$

where ω_n are the discrete eigenvalues of the operator with degeneracies d_n . The analytic structure of the ζ -function (1.2) (for the Laplacian acting on scalar fields) was first studied by Minakshisundaram and Pleijel [26]. The sum usually converges for $\text{Re } z > N/2$, where N is the dimension of the compact manifold M , and is determined for the other values of z by analytic continuation in z .

It is well known that the N -sphere (S^N) is the Euclidean section for N -dimensional de Sitter spacetime. One-loop calculations have been performed using the well-known spectrum of the Laplacian on S^N [1,13].

It has been pointed out [3,6] that the Euclidean section for anti-de Sitter spacetime is the real hyperbolic space H^N , the noncompact Riemannian symmetric space (SS) of rank one which is "dual" to S^N in the sense of SS theory (see Ref. [18]). For noncompact manifolds the definition of the ζ -function is more complicated. The compact ζ -function (1.2) can be obtained as a Mellin transform of the trace of the heat kernel

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr } K(t), \quad (1.3)$$

where Tr means both a trace over the spinor-tensor indices in the heat kernel $K_{ab}(x, x', t)$,² and an integration over the manifold:

$$\text{Tr } K(t) = \int_M dx \sum_a K_a^a(x, x, t). \quad (1.4)$$

² Here the spinor-tensor indices at x and x' are collectively represented by a and b , respectively.

In the noncompact case we can only define a “local” ζ -function by

$$\zeta(z, x) = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \sum_a K_a^a(x, x, t), \tag{1.5}$$

the integral of this quantity over the manifold being (generally) divergent. If M is a homogeneous space G/K , where G and $K (\subset G)$ are group spaces, the scalar function $\sum_a K_a^a(x, x, t)$ is independent of $x \in M$ and equals $\sum_a K_a^a(x_0, x_0, t)$, where x_0 is the “origin” of G/K (which may be chosen arbitrarily). In this case the right-hand side of Eq. (1.5) defines a function of z only, denoted by $\zeta(z)$. On the hyperbolic space H^N , which is homogeneous being isomorphic to the coset space $SO_0(N, 1)/SO(N)$, the ζ -function (1.5) takes the form³

$$\zeta(z) \propto \int_0^\infty d\lambda \mu(\lambda) \omega_\lambda^{-z}, \tag{1.6}$$

where the real parameter λ labels the continuous spectrum of the Laplacian and the ω_λ are the corresponding eigenvalues. In the case of a noncompact Euclidean section, the function $\mu(\lambda)$ plays the same role as the discrete degeneracies d_n in Eq. (1.2).

For scalar fields the function $\mu(\lambda)$ is known in the mathematical literature as the *Plancherel measure* [19], because it arises in the inversion formula for the spherical transform and in the Plancherel formula on a noncompact Riemannian SS G/K . For example on H^N these formulae are

$$\hat{f}(\lambda) \equiv \int_0^\infty f(y) \phi_\lambda(y) (\sinh y)^{N-1} dy, \tag{1.7}$$

$$f(y) = c_N \int_0^\infty \hat{f}(\lambda) \phi_\lambda(y) \mu(\lambda) d\lambda, \tag{1.8}$$

³Strictly speaking Eq. (1.6) gives only the continuous part of the zeta function. For N even the Laplacian on H^N may also have a discrete spectrum, due to the existence of discrete series for the group $SO_0(N, 1)$. (These are the irreducible unitary representations whose matrix elements are square-integrable.) It is known that the discrete series do not appear in the Plancherel formula for scalars [20], spinors [5] or symmetric transverse traceless tensors [7] on H^N ($N \geq 3$). (The scalar result generalizes to arbitrary noncompact Riemannian SS G/K , since the trivial representation of K is never contained in the discrete series, see [25, p. 455].) However it is clear that the discrete series will eventually enter in the Plancherel formula for a generic vector bundle over H^N . It will be seen here (see also [10]) that for p -forms on H^N the spectrum of Δ is purely continuous except for N even and $p = \frac{1}{2}N$. In this case there are square-integrable p -forms which are harmonic and contribute a discrete part to the spectrum. We shall often refer to $\mu(\lambda)$ as the “Plancherel measure”, meaning the continuous part of it. The discrete part of the Plancherel measure is just the formal degree of the discrete series (e.g. Ref. [37, vol. II, p. 407]) and will be calculated in Section 6.

$$\int_0^\infty |f(y)|^2 (\sinh y)^{N-1} dy = c_N \int_0^\infty |\hat{f}(\lambda)|^2 \mu(\lambda) d\lambda. \quad (1.9)$$

Here $f(y)$ is a *zonal* (i.e. $SO(N)$ -invariant) function on H^N with compact support, which depends only on the geodesic distance y from the origin, and $\phi_\lambda(y)$ are the (scalar) spherical functions, i.e., the eigenfunctions of the radial Laplacian which satisfy $\phi_\lambda(0) = 1$ (see, e.g., Ref. [19]). The constant c_N is given by

$$c_N = 2^{N-2}/\pi. \quad (1.10)$$

The functions $\phi_\lambda(y)$ satisfy the orthogonality relations

$$\int_0^\infty \phi_\lambda(y) \phi_{\lambda'}(y) (\sinh y)^{N-1} dy = \frac{\delta(\lambda - \lambda')}{c_N \mu(\lambda)}, \quad (1.11)$$

$$\int_0^\infty \phi_\lambda(y) \phi_\lambda(y') \mu(\lambda) d\lambda = \frac{\delta(y - y')}{c_N \sinh^{N-1} y}, \quad (1.12)$$

where δ is the Dirac distribution. The function $\mu(\lambda)$ (for scalars on H^N) is given by [3]

$$\mu(\lambda) = \frac{\pi}{\left[2^{N-2} \Gamma(\frac{1}{2}N)\right]^2} \left| \frac{\Gamma(i\lambda + \frac{1}{2}(N-1))}{\Gamma(i\lambda)} \right|^2. \quad (1.13)$$

More generally, the scalar Plancherel measure is known on any Riemannian symmetric space of the noncompact type (with negative curvature), and is given by

$$\mu(\lambda) = [C(\lambda)C(-\lambda)]^{-1}, \quad (1.14)$$

where the function $C(\lambda)$, known as the Harish–Chandra function, is given in terms of a product over the positive roots of the symmetric space (see, e.g., Eq. (5.38) of Ref. [4]). (If the rank of the symmetric space is l (≥ 1), the spectrum label λ is a vector with l components, $\lambda \in \mathcal{A}$, where $\mathcal{A} \simeq \mathbb{R}^l$ is a Cartan subspace of the symmetric space [4].) The function $C(\lambda)$ is related to the asymptotic form at infinity of the spherical functions. Recently the concept of spherical functions has been generalized to Dirac spinors on H^N , and $\mu(\lambda)$ has been calculated in that case [5].

The Plancherel measure can be generalized naturally to arbitrary fields (vector, tensor, spinor, etc.) on a SS. It is different from the scalar Plancherel measure in general. In this paper the Plancherel measure will be calculated for antisymmetric tensor fields (p -forms) on H^N . We shall first construct the eigenmodes of the Hodge–de Rham operator by working in geodesic polar

coordinates. We shall do this for S^N , in Sections 2 and 3, and rederive the spectrum of Δ on S^N (first calculated in Refs. [14,23,24]) independently. By analytic continuation we shall obtain, in Section 4, the eigenmodes on H^N , and compute their normalization factors to find the continuous part $\mu(\lambda)$ of the Plancherel measure. It is enough to do the calculation for coexact p -eigenforms, the result for the exact ones following by a simple transformation (see Section 2).

The function $\mu(\lambda)$, obtained in this way, has the following properties. For N odd it is an analytic function and in fact it reduces to a polynomial in the variable λ^2 . For N even $\mu(\lambda)$ can be continued to a meromorphic function in the complex λ -plane, with simple poles on the imaginary axis. The following relation will be demonstrated:

$$\frac{\mu(i(\rho + n))}{\mu(i(\rho + 1))} = \frac{d_n}{d_1}, \quad n = 1, 2, \dots, \tag{1.15}$$

where $\rho \equiv (N - 1)/2$, d_n are the degeneracies of Δ acting on p -forms on S^N , and where it is understood that for N even the left-hand side means the ratio of the residues of $\mu(\lambda)$ at the poles $\lambda_n = i(\rho + n)$ and $\lambda_1 = i(\rho + 1)$. This is a generalization of the results obtained in Refs. [4] and [20] for scalar fields. A similar result has been obtained also for spinor fields [5] and for symmetric transverse traceless tensor fields [7] on H^N .

The discrete part of the spectrum is considered in Section 5. We construct square-integrable harmonic k -forms on H^{2k} , whose existence was demonstrated by Dodziuk [11].

In Section 6 we give an alternative derivation of $\mu(\lambda)$ using group-theoretic methods. Given a vector bundle E^τ over G/K (determined by the irreducible representation (irrep) τ of K), the space $L^2(G/K, E^\tau)$ of square-integrable sections of E^τ can be regarded as a subspace of $L^2(G)$ (the space of square-integrable functions on G) in a natural way. In fact the regular representation π of G on $L^2(G)$ is unitarily equivalent to the direct sum over the irreps τ of K of the induced representations π_τ of G on $L^2(G/K, E^\tau)$ (each π_τ being counted a number of times equal to the dimension of τ), see e.g. Ref. [27]. It follows that the Plancherel measure for an arbitrary vector bundle over G/K can be determined from the Plancherel measure for scalar functions on the group G . One simply needs to identify the irreducible unitary representations in the Plancherel formula for $L^2(G)$ (discrete series and principal series) whose restriction to K contains the given τ (see Ref. [27, Lemma 1]). The advantage of this method is that the Plancherel formula for $L^2(G)$ has been worked out by Harish-Chandra for general semisimple groups (see e.g. Ref. [25]). For the Lorentz group $SO_0(N, 1)$ the formula takes a simple form, which has been given, e.g., by Hirai [22] (see also [31,28] for $N = 4$). Using Hirai's formula we reobtain the results of Sections 4 and 5. This method can be used to obtain the Plancherel measure for arbitrary fields on H^N and prove a result analogous

to (1.15).

In Section 7 we examine the analytic properties of the zeta functions of Δ on H^N and S^N . We find that on both spaces $\zeta(z)$ extends to a meromorphic function in the complex z -plane with simple poles at $z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, 1$ for N even, and $z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, -\infty$ for N odd and $p \neq \rho$. For $p = \rho$, the ζ -function on H^N is undefined while that on S^N has only a finite number of poles at $z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, \frac{1}{2}$. The zeta functions have so-called “trivial” zeros at the negative integers in the odd-dimensional case. In Section 8 we obtain a relation between the p -form zeta functions on H^N and S^N . We obtain a contour representation of the ζ -function on the even-dimensional sphere by deforming the contour of integration of (1.6) in the complex λ -plane and using the relation (1.15) between the residues of $\mu(\lambda)$ and the degeneracies d_n on S^N . A variation of this idea is then used to find similar results in the odd-dimensional case.

2. Exact and coexact p -forms on the N -sphere

We assume that the reader is familiar with the theory of p -forms on manifolds, and with the basic properties of the differential operators acting on them, i.e., the exterior derivative d , the coderivative δ , and the Hodge–de Rham Laplacian $\Delta = d\delta + \delta d$ (see, e.g., Refs. [36,9]).

From the Hodge decomposition theorem (e.g. Ref. [36, p. 233]) every p -form ω on a compact Riemannian N -dimensional manifold M ($p = 0, 1, \dots, N$) can be uniquely written as the sum

$$\omega = d\alpha + \delta\beta + \omega_h \quad (2.1)$$

of an exact form ($d\alpha$), a coexact form ($\delta\beta$), and a harmonic form ($\Delta\omega_h = 0$). It is well known that each de Rham cohomology class on a compact manifold contains a unique harmonic representative. Thus the space of harmonic p -forms is isomorphic to the cohomology space $H^p(M) = \text{closed } p\text{-forms/exact } p\text{-forms}$, i.e. $b_p \equiv \dim H^p(M) = \dim \text{Ker } \Delta_p$ (see Ref. [36, p. 225] or Ref. [9, p. 400]). On the N -sphere, $b_0 = b_N = 1$ and $b_p = 0$ for $1 \leq p \leq N - 1$ [9]. Therefore for $1 \leq p \leq N - 1$ there are no harmonic p -forms on S^N , and any p -form can be written as $d\alpha + \delta\beta$ and is exact (coexact) if and only if it is closed (coclosed).

Now let ω be an eigenform of Δ with eigenvalue λ , i.e. $\Delta\omega = \lambda\omega$. Since d and δ both commute with Δ , it follows that $d\omega$ and $\delta\omega$ are also eigenforms of Δ with the same eigenvalue. In particular, if $d\alpha$ is an exact p -eigenform of Δ , $\beta = \delta d\alpha$ is a coexact $(p - 1)$ -eigenform of Δ with the same eigenvalue. Therefore if $\omega_n^E(p)$ [$\omega_n^{\text{CE}}(p)$] and $d_n^E(p)$ [$d_n^{\text{CE}}(p)$] denote the eigenvalues and the degeneracies of Δ acting on exact (coexact) p -forms, we have for $p = 1, 2, \dots, N$

$$\omega_n^E(p) = \omega_n^{CE}(p - 1), \quad d_n^E(p) = d_n^{CE}(p - 1). \tag{2.2}$$

For this reason we shall consider only coexact (i.e. divergenceless) p -forms in the following. Another relation between eigenvalues and degeneracies for $p = 0, 1, \dots, N - 1$ is

$$\omega_n^{CE}(p) = \omega_n^{CE}(N - p - 1), \quad d_n^{CE}(p) = d_n^{CE}(N - p - 1). \tag{2.3}$$

To see this remember that the dual operator \star maps a p -form to an $(N - p)$ -form and satisfies $\star\star = (-1)^{p(N-p)}$ [36]. From the definition $\delta = (-1)^p \star^{-1} d \star$ it follows that the dual of a coexact form is an exact form. Since \star commutes with Δ , the dual of a coexact p -eigenform is an exact $(N - p)$ -eigenform with the same eigenvalue. Therefore $d_n^{CE}(p) = d_n^E(N - p)$. (Notice that for N even and $p = N/2$ exact and coexact forms have the same spectrum.) The result (2.3) then follows from (2.2). Since the case with $p = 0$ reduces to the scalar theory, we shall assume that $1 \leq p \leq N - 1$ unless stated otherwise.

Now let H be a coexact p -eigenform of Δ satisfying

$$\delta H = 0, \tag{2.4}$$

$$\delta d H = \lambda H. \tag{2.5}$$

In local coordinates these equations read

$$\nabla^\alpha H_{\alpha\mu_1\dots\mu_{p-1}} = 0, \tag{2.6}$$

$$-(p + 1)\nabla^\alpha \nabla_{[\alpha} H_{\mu_1\dots\mu_p]} = (L + p)(L + N - p - 1)H_{\mu_1\dots\mu_p}, \tag{2.7}$$

where ∇_α denotes the covariant derivative with respect to the Levi-Civita connection of the canonical metric on S^N (of unit radius), and we have set $\lambda = (L + p)(L + N - p - 1)$ with L to be determined. (It will be shown that regularity of the eigenmodes requires $L = 1, 2, \dots, \infty$.)

It is known (see, e.g., the appendix by Dodziuk in Ref. [8]) that on a compact manifold \mathcal{A} is a positive definite elliptic operator with discrete spectrum. [Remember that on scalar functions $\Delta f = -\nabla_\alpha \nabla^\alpha f$.] The spectrum of Δ on S^N has been calculated in Refs. [14,23,24]. We give (below and in Section 3) an independent derivation based on the evaluation of the eigenmodes in geodesic polar coordinates. We write the metric on S^N as

$$ds^2 = d\chi^2 + \sin^2 \chi d\theta_{N-1}^2, \tag{2.8}$$

where $d\theta_{N-1}^2$ is the line element of S^{N-1} . Define the rescaled forms $F_{i_1\dots i_{p-1}}^{(1)}$ and $F_{i_1\dots i_p}^{(2)}$ on S^{N-1} (depending on χ as a parameter) by

$$H_{\chi i_1\dots i_{p-1}} \equiv (\sin \chi)^{p-2} F_{i_1\dots i_{p-1}}^{(1)}, \tag{2.9}$$

$$H_{i_1\dots i_p} \equiv (\sin \chi)^p F_{i_1\dots i_p}^{(2)}. \tag{2.10}$$

Then we can rewrite Eq. (2.6) as

$$\delta F^{(1)} = 0, \quad (2.11)$$

$$\left[\frac{\partial}{\partial \chi} + (N - p - 1) \cot \chi \right] F^{(1)} - \delta F^{(2)} = 0, \quad (2.12)$$

where d and δ acting on $F^{(1)}$ and $F^{(2)}$ are the exterior differential and codifferential operators on S^{N-1} . Eq. (2.7) can be written as

$$\left[\frac{\partial^2}{\partial \chi^2} + (N - 1) \cot \chi \frac{\partial}{\partial \chi} + \frac{1}{\sin^2 \chi} \{-\delta d + (p - 1)(N - p - 1)\} \right] F^{(1)} = -L(L + N - 1)F^{(1)} \quad (2.13)$$

(we have used the condition (2.12) to derive this), and

$$\left[\frac{\partial^2}{\partial \chi^2} + (N - 1) \cot \chi \frac{\partial}{\partial \chi} + \frac{1}{\sin^2 \chi} \{-\delta d + p(N - p - 2)\} \right] F^{(2)} - \frac{1}{\sin^2 \chi} \left[\frac{\partial}{\partial \chi} + (N - p - 3) \cot \chi \right] dF^{(1)} = -L(L + N - 1)F^{(2)}. \quad (2.14)$$

The solutions of Eq. (2.13) are

$$F^{(1)(Ll\sigma)} = Q_{Ll}(\chi) \tilde{H}^{(l\sigma)}, \quad (2.15)$$

where $\tilde{H}^{(l\sigma)}$ is a χ -independent coexact $(p - 1)$ -form on S^{N-1} which satisfies an equation analogous to (2.7) (with label l in place of L), i.e.

$$\delta d \tilde{H}^{(l\sigma)} = (l + p - 1)(l + N - p - 1) \tilde{H}^{(l\sigma)}, \quad (2.16)$$

and $Q_{Ll}(\chi)$ is given by

$$Q_{Ll}(\chi) \equiv (\sin \chi)^l F(L + N + l - 1, l - L; l + \frac{1}{2}N; \sin^2 \frac{1}{2}\chi), \quad (2.17)$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. In Eq. (2.16) σ represents the labels other than L and l . The condition (2.12) is solved as

$$F^{(2)(Ll\sigma)} = \frac{1}{(l + p - 1)(l + N - p - 1)} \times \left[\frac{d}{d\chi} + (N - p - 1) \cot \chi \right] Q_{Ll}(\chi) d\tilde{H}^{(l\sigma)}. \quad (2.18)$$

One can show that this satisfies (2.14) as well. When $F^{(1)} = 0$, the solutions of (2.14) are simply

$$F^{(2)} = Q_{Ll}(\chi) \tilde{H}, \quad (2.19)$$

where \tilde{H} is a coexact p -form on S^{N-1} satisfying

$$\delta d \tilde{H} = (l + p)(l + N - p - 2) \tilde{H}. \quad (2.20)$$

Now let us show that $L = 1, 2, \dots, \infty$. The proof is essentially by induction over the dimension N . Assume that $l = 1, 2, \dots, \infty$ for $1 \leq p \leq N - 2$, and recall that $l = 0, 1, \dots, \infty$ for $p = 0$. (Obviously, there are no coexact $(N - 1)$ -forms on S^{N-1} .) Then, one finds from (2.15) that L must be an integer larger than or equal to l for the regularity of these solutions at $\chi = \pi$. This shows that $L = 1, 2, \dots, \infty$ for $p \geq 2$, and for L fixed $l = 1, 2, \dots, L$. For $p = 1$, there is no solution with $L = l = 0$ because Eq. (2.12) cannot be solved if $F^{(1)} = \text{constant}$.

3. Degeneracies on S^N

For the solutions given by (2.15) and (2.18) we find

$$\begin{aligned} \langle H^{(Ll\sigma)}, H^{(Ll\sigma)} \rangle &\equiv \int_{S^N} H^{(Ll\sigma)*} \wedge \star H^{(Ll\sigma)} \\ &= \frac{p(L+p)(L+N-p-1)}{(l+p-1)(l+N-p-1)} \int d\chi \sin^{N-1} \chi |Q_{Ll}(\chi)|^2, \end{aligned} \tag{3.1}$$

where

$$H^* \wedge \star H \equiv H_{\mu_1 \dots \mu_p}^* H^{\mu_1 \dots \mu_p} d\Omega_N, \tag{3.2}$$

and $d\Omega_N$ is the volume form on S^N . (The asterisk indicates complex conjugation.) We have assumed that the $(p - 1)$ -form $\tilde{H}^{(l\sigma)}$ on S^{N-1} is normalized by $\langle \tilde{H}^{(l\sigma)}, \tilde{H}^{(l\sigma)} \rangle = 1$. Let the normalized eigenforms $\hat{H}^{(Ll\sigma)} = \mathcal{N}_{Ll} H^{(Ll\sigma)}$ satisfy $\langle \hat{H}^{(Ll\sigma)}, \hat{H}^{(Ll\sigma)} \rangle = 1$. The normalization factor \mathcal{N}_{Ll} can be found from (3.1) as

$$\mathcal{N}_{Ll} = \left[\frac{(l+p-1)(l+N-p-1)}{p(L+p)(L+N-p-1)} \right]^{1/2} N_{Ll}, \tag{3.3}$$

where N_{Ll} is the normalization factor for the scalar case [7] given by

$$\begin{aligned} N_{Ll} &= \left[2^{l+(N-2)/2} \Gamma(l + \frac{1}{2}N) \right]^{-1} \\ &\times \left[\frac{2L+N-1}{2} \frac{(L+l+N-2)!}{(L-l)!} \right]^{1/2}. \end{aligned} \tag{3.4}$$

The degeneracies are given by

$$D_N(L, p) = \sum_{L \text{ fixed}} \int_{S^N} d\Omega_N \hat{H}^* \cdot \hat{H} = \Omega_N \lim_{\chi \rightarrow 0} \sum_{L \text{ fixed}} \hat{H}^* \cdot \hat{H}, \tag{3.5}$$

where $H^{(1)} \cdot H^{(2)} \equiv H_{\mu_1 \dots \mu_p}^{(1)} H^{(2)\mu_1 \dots \mu_p}$, and where the summation is over all the normalized solutions \hat{H} with the same ‘‘angular momentum’’ label L . We have used the fact that $\sum_{L \text{ fixed}} \hat{H}^* \cdot \hat{H}$ is constant over S^N . The volume of S^N is

$$\Omega_N = 2\pi^{(N+1)/2} / \Gamma(\frac{1}{2}(N + 1)). \tag{3.6}$$

Now, we find from the rotational symmetry of $\sum_{L \text{ fixed}} \hat{H}^* \otimes \hat{H}$,

$$\lim_{\chi \rightarrow 0} \sum_{L \text{ fixed}} \hat{H}^* \cdot \hat{H} = N \lim_{\chi \rightarrow 0} \sum_{l\sigma} \hat{H}_{\chi^{i_1 \dots i_{p-1}}}^{(L\sigma)*} \hat{H}^{(L\sigma)\chi^{i_1 \dots i_{p-1}}}. \tag{3.7}$$

We find from (2.15) that $H_{\chi^{i_1 \dots i_{p-1}}}^{(L\sigma)*} H^{(L\sigma)\chi^{i_1 \dots i_{p-1}}} \rightarrow 0$ for $\chi \rightarrow 0$ if $l \neq 1$. Hence

$$D_N(L, p) = N \frac{\Omega_N}{\Omega_{N-1}} |\mathcal{N}_{L1}|^2 D_{N-1}(1, p - 1), \tag{3.8}$$

where we have used $(\sin \chi)^{-l} Q_{Ll}(\chi) \rightarrow 1$ for $\chi \rightarrow 0$ and

$$\Omega_{N-1} \sum_{\sigma} \tilde{H}^{(l\sigma)*} \cdot \tilde{H}^{(l\sigma)} = D_{N-1}(l, p - 1). \tag{3.9}$$

An explicit evaluation of (3.8) using (3.3) and (3.6) leads to

$$D_N(L, p) = \frac{(N - p)(2L + N - 1)(L + N - 1)!}{(L + p)(L + N - p - 1)N!(L - 1)!} D_{N-1}(1, p - 1). \tag{3.10}$$

For $L = 1$ we have

$$D_N(1, p) = \frac{N + 1}{p + 1} D_{N-1}(1, p - 1) = \frac{(N + 1)!}{(p + 1)!(N - p)!}, \tag{3.11}$$

where we have used $D_N(1, 0) = N + 1$. Hence we obtain

$$D_N(L, p) = \frac{(2L + N - 1)(L + N - 1)!}{p!(N - p - 1)!(L - 1)!(L + p)(L + N - p - 1)}, \tag{3.12}$$

in agreement with Refs. [14,23,24].

4. The Plancherel measure on H^N

The ζ -function of the Hodge-de Rham operator acting on coclosed p -forms on H^N is defined as the Mellin transform of the trace of the heat kernel

$$\zeta^{(H)}(z) \equiv \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} K_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x, t). \tag{4.1}$$

Here the heat kernel is defined by

$$(\partial/\partial t + \Delta_x) K_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x', t) = 0, \tag{4.2}$$

$$K_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x', 0) = \delta_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x'), \tag{4.3}$$

where $\delta_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x')$ is the integral kernel for the projection operator onto the space of coclosed p -forms. Homogeneity of the hyperbolic space implies that the right-hand side of (4.1) is indeed a function of z only.

Now let the coclosed p -eigenform $\hat{h}^{(\lambda u)}(x)$ satisfy

$$\Delta \hat{h}^{(\lambda u)} = [\lambda^2 + (\rho - p)^2] \hat{h}^{(\lambda u)}, \tag{4.4}$$

where u is the discrete label for distinguishing eigenforms with the same eigenvalue, and where $\rho \equiv (N - 1)/2$. As we shall see, λ is real and can be taken to be positive for the continuous spectrum (see also [10]). It can easily be seen from $\Delta = \delta d + d \delta$ that a coclosed p -eigenform with positive λ is coexact. (For this reason we shall often call the Plancherel measure obtained in this section the Plancherel measure for coexact forms.) Let these eigenmodes be normalized as

$$\langle \hat{h}^{(\lambda u)}, \hat{h}^{(\lambda' u')} \rangle \equiv \int_{H^N} \hat{h}^{(\lambda u)*} \wedge \star \hat{h}^{(\lambda' u')} = \delta_{uu'} \delta(\lambda - \lambda'). \tag{4.5}$$

The heat kernel can be expressed in terms of $\hat{h}_{\mu_1 \dots \mu_p}^{(\lambda u)}(x)$ as

$$\begin{aligned} K_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_p}(x, x', t) \\ = \sum_u \int_0^\infty d\lambda \hat{h}_{\mu_1 \dots \mu_p}^{(\lambda u)}(x) \hat{h}^{(\lambda u) \nu_1 \dots \nu_p}(x')^* e^{-[\lambda^2 + (\rho - p)^2]t}, \end{aligned} \tag{4.6}$$

if there are no discrete eigenvalues. This expression must be modified for N even and $p = \frac{1}{2}N$ due to the existence of square-integrable harmonic forms (see the next section).

By using the mode expansion (4.6) in (4.1) we find

$$\zeta^{(H)}(z) = \frac{c_N}{\Omega_{N-1}} g(p) \int_0^\infty d\lambda \frac{\mu(\lambda)}{[\lambda^2 + (\rho - p)^2]^z}, \tag{4.7}$$

where the Plancherel measure $\mu(\lambda)$ is defined by

$$\mu(\lambda) \equiv \frac{\Omega_{N-1}}{c_N g(p)} \sum_u \hat{h}^{(\lambda u)*} \cdot \hat{h}^{(\lambda u)}(0). \tag{4.8}$$

[Eq. (4.7) becomes undefined for N odd and $p = \rho$, but Eq. (4.8) still serves as the definition of $\mu(\lambda)$.] The factor Ω_{N-1} is the volume of S^{N-1} given by (3.6) with $N \rightarrow N - 1$, and the “spin factor” $g(p)$ is the number of independent components of a coexact p -form given by

$$g(p) = \frac{(N - 1)!}{p!(N - p - 1)!}. \tag{4.9}$$

Unnormalized solutions $h^{(\lambda u)}(x)$ can be obtained from the results in Section 2 by letting $\chi = iy$ and $L = -\rho + i\lambda$, where y is the geodesic distance from the origin on H^N . We need only the eigenforms with $\hat{h}_{y_{i_1} \dots i_{p-1}}^{(\lambda u)*}(0) \hat{h}^{(\lambda u) y_{i_1} \dots i_{p-1}}(0) \neq$

0 to determine $\mu(\lambda)$ from (4.8). This is because the rotational symmetry around the origin leads to

$$\sum_u \hat{h}^{(\lambda u)*} \cdot \hat{h}^{(\lambda u)} = N \sum_u \hat{h}_{y_{i_1 \dots i_{p-1}}}^{(\lambda u)}(0)^* \hat{h}^{(\lambda u) y_{i_1 \dots i_{p-1}}}(0). \tag{4.10}$$

These p -eigenforms are obtained by analytically continuing the eigenforms on S^N given by (2.15) and (2.18) with $l = 1$. To determine $\hat{h}_{y_{i_1 \dots i_{p-1}}}^{(\lambda \sigma)}$ completely, we need to evaluate the normalization factor. Define $\hat{h}_{\mu_1 \dots \mu_p}^{(\lambda \sigma)} \equiv \mathcal{N}_{\lambda l} h_{\mu_1 \dots \mu_p}^{(\lambda \sigma)}$. The normalization integral for the unnormalized p -eigenforms $h^{(\lambda u)}(x)$ can be turned into a surface integral [7]

$$\begin{aligned} \langle h^{(\lambda u)}, h^{(\lambda' u')} \rangle &= \lim_{A \rightarrow \infty} \frac{\sinh^{N-1} y}{\lambda^2 - \lambda'^2} \\ &\times \int d\Omega_{N-1} \left(h^{(\lambda u)*} \cdot \frac{\partial}{\partial y} h^{(\lambda' u')} - \frac{\partial}{\partial y} h^{(\lambda u)*} \cdot h^{(\lambda' u')} \right) \Big|_{y=A}. \end{aligned} \tag{4.11}$$

We find that only the $h_{i_1 \dots i_p}^{(\lambda u)}$ components contribute. The large- y behavior of $h_{i_1 \dots i_p}^{(\lambda \sigma)}$ can be found from (2.18) and

$$q_{\lambda l}(y) \equiv Q_{(-\rho+i\lambda)l}(iy) \approx i^l \left[c_l(\lambda) e^{(-\rho+i\lambda)y} + (\lambda - \lambda) \right], \tag{4.12}$$

where

$$c_l(\lambda) = \frac{2^{N+l-2} \Gamma(l + \frac{1}{2}N) \Gamma(i\lambda)}{\sqrt{\pi} \Gamma(i\lambda + \frac{1}{2}(N-1) + l)} \tag{4.13}$$

(see Ref. [7]). We find

$$\begin{aligned} h_{i_1 \dots i_p}^{(\lambda \sigma)} &\approx \frac{i^{l+p} (\sinh y)^p}{(l+p-1)(l+N-p-1)} \\ &\times \left[(i\lambda + \rho - p) c_l(\lambda) e^{(-\rho+i\lambda)y} p \nabla_{[i_1} \hat{H}_{i_2 \dots i_p]}^{(\lambda \sigma)} + (\lambda - \lambda) \right] \end{aligned} \tag{4.14}$$

unless $i\lambda = \pm(p - \rho)$ with $p \geq \rho$. From this we find that λ must be real or $\mp i(p - \rho)$ for the p -eigenform to be normalizable. Here we let λ be real and positive since we are concerned with the continuous spectrum. (Note that the replacement $\lambda \rightarrow -\lambda$ leaves the eigenform unchanged.) We shall come back to the case with imaginary values of λ in the next section.

Substituting (4.14) in (4.11), we obtain up to a phase factor

$$\mathcal{N}_{\lambda l} = \frac{\sqrt{c_N(l+p-1)(l+N-p-1)}}{\sqrt{p} [\lambda^2 + (p-\rho)^2]^{1/2} c_l(\lambda)}. \tag{4.15}$$

Then, the Plancherel measure $\mu(\lambda)$ for coexact p -forms is obtained by the procedure we used for the degeneracies as

$$\mu(\lambda) = \frac{N}{c_N g(p)} |\mathcal{N}_{\lambda 1}|^2 D_{N-1}(1, p-1) \tag{4.16}$$

$$= \frac{\pi}{\left[2^{N-2} \Gamma\left(\frac{1}{2}N\right)\right]^2 [\lambda^2 + (\rho - p)^2]} \left| \frac{\Gamma(i\lambda + \rho + 1)}{\Gamma(i\lambda)} \right|^2. \tag{4.17}$$

Written explicitly Eq. (4.17) reads

$$\mu(\lambda) = \frac{\pi}{\left[2^{N-2} \Gamma\left(\frac{1}{2}N\right)\right]^2 [\lambda^2 + (\rho - p)^2]} \prod_{j=0}^{\rho} (\lambda^2 + j^2), \quad N \text{ odd}, \tag{4.18}$$

$$\mu(\lambda) = \frac{\pi \lambda \tanh(\pi \lambda)}{\left[2^{N-2} \Gamma\left(\frac{1}{2}N\right)\right]^2 [\lambda^2 + (\rho - p)^2]} \prod_{j=1/2}^{\rho} (\lambda^2 + j^2), \quad N \text{ even}. \tag{4.19}$$

For $p = 0$ and $p = 1$ these coincide with the known expressions of $\mu(\lambda)$ for scalars [3] and divergenceless vectors [7], respectively. The Plancherel measure for exact p -forms is obtained by letting $p \rightarrow p - 1$ in these formulae.

For N odd, $\mu(\lambda)$ is clearly a polynomial in λ^2 (the apparent poles at $\lambda = \pm i|\rho - p|$ are eliminated by the corresponding factor in the product over j). For N even, $\mu(\lambda)$ can be continued to a meromorphic function in the complex λ -plane, with simple poles on the imaginary axis at

$$\lambda = \pm i|\rho - p|, \pm i(\rho + 1), \pm i(\rho + 2), \dots, \pm i\infty. \tag{4.20}$$

We can now easily verify the following relation between the degeneracies (3.12) of Δ on S^N and the p -form Plancherel measure $\mu(\lambda)$ on H^N :

Theorem 4.1. *The Plancherel measure $\mu(\lambda)$ on H^N satisfies*

$$\frac{\mu(i(L + \rho))}{\mu(i(1 + \rho))} = \frac{D_N(L, p)}{D_N(1, p)}, \quad L = 1, 2, \dots, \tag{4.21}$$

where for N even the left-hand side means the ratio of the residues of $\mu(\lambda)$ at the given points.

Proof. Using (4.17) for $\mu(\lambda)$ we see that the factor $[\lambda^2 + (\rho - p)^2]^{-1}$ gives, at $\lambda = i(L + \rho)$, the last two terms in the denominator of $D_N(L, p)$ in (3.12). The remaining (L -dependent) terms in (3.12) arise from calculating the ratio of Γ -functions in (4.17) at $\lambda = i(L + \rho)$ and $\lambda = i(1 + \rho)$, and substituting on the left-hand side of (4.21). The proof for the exact p -forms follows simply by letting $p \rightarrow p - 1$ and repeating the same argument. \square

This result is analogous to those obtained in Refs. [5,7] for Dirac spinors and for the symmetric transverse traceless tensor fields on H^N .

5. Square-integrable harmonic forms

The coclosed forms on H^N which are obtained by analytic continuation from the p -eigenforms on S^N with $F^{(1)} = 0$ in Section 2 [see (2.19)] cannot be square-integrable. This is because the normalization integral reduces to that for the scalar field in this case. We have seen in the previous section that square-integrable coclosed p -eigenforms do not exist unless $i\lambda = p - \rho \geq 0$. This implies that $\Delta h = 0$, i.e., a square-integrable coclosed form must be harmonic. Further examination of the asymptotic behavior of these harmonic forms reveals that the dimension N must be even and $p = \frac{1}{2}N$. Then, we find that these forms are indeed square-integrable. They are given as follows (compare to Thm. 4.4 of Ref. [10]):

$$\hat{h}_{y i_1 \dots i_{N/2-1}} = \left[\frac{2l + N - 2}{N} \right]^{1/2} \frac{1}{\sinh y} \left(\tanh \frac{1}{2} y \right)^{l + N/2 - 1} \tilde{H}_{i_1 \dots i_{N/2-1}}, \quad (5.1)$$

$$\hat{h}_{i_1 \dots i_{N/2}} = \left[\frac{N}{2l + N - 2} \right]^{1/2} \left(\tanh \frac{1}{2} y \right)^{l + N/2 - 1} \tilde{\nabla}_{[i_1} \tilde{H}_{i_2 \dots i_{N/2}]}, \quad (5.2)$$

where \tilde{H} is an $(\frac{1}{2}N - 1)$ -form on S^{N-1} satisfying $\Delta \tilde{H} = (l + N/2 - 1)^2 \tilde{H}$, and $l = 1, 2, \dots, \infty$. The normalization factor has been determined by requiring $\langle \hat{h}, \hat{h} \rangle = 1$. The form \hat{h} can be reexpressed in terms of a harmonic $(\frac{1}{2}N - 1)$ -form as

$$\hat{h} = dA, \quad (5.3)$$

where

$$A_{y i_1 \dots i_{N/2-2}} = 0, \quad (5.4)$$

$$A_{i_1 \dots i_{N/2-1}} = \frac{2}{\sqrt{N(2l + N - 2)}} \left(\tanh \frac{1}{2} y \right)^{l + N/2 - 1} \tilde{H}_{i_1 \dots i_{N/2-1}}. \quad (5.5)$$

The square-integrable harmonic $\frac{1}{2}N$ -forms are both closed and coclosed. This result agrees with a general theorem of Andreotti and Vesentini (see [11]). Note that the harmonic $(\frac{1}{2}N - 1)$ -form A is not square-integrable. There are other linearly independent harmonic $(\frac{1}{2}N - 1)$ -forms (with $A_{y i_1 \dots i_{N/2-2}} \neq 0$), but they are exact themselves and, therefore, give zero when substituted in (5.3).

Note that the harmonic forms given by (5.1) and (5.2) constitute a reducible representation of $SO_0(N, 1)$ because one can require $\star \hat{h} = \pm i^{N/2} \hat{h}$. Define

$$\tilde{\star} \tilde{H} \equiv \frac{i^{N/2}}{l + N/2 - 1} \star d \tilde{H}. \quad (5.6)$$

Then we find that $\star\hat{h} = \pm i^{N/2}\hat{h}$ if and only if $\tilde{\star}\tilde{H} = \pm\tilde{H}$. In Section 6 this reducible representation will be identified as the direct sum of two irreducible representations in the “discrete series” of $SO_0(N, 1)$.

There is a discrete contribution to the Plancherel measure for N even and $p = \frac{N}{2}$ due to the square-integrable harmonic forms. For example, the traced heat kernel for coclosed $\frac{N}{2}$ -forms on H^N is

$$\text{Tr } K(t) = \frac{c_{Ng}(\frac{1}{2}N)}{\Omega_{N-1}} \left(\int_0^\infty d\lambda \mu_{N/2}(\lambda) e^{-(\lambda^2+1/4)t} + \mu_0 \right), \tag{5.7}$$

where $\mu_{N/2}(\lambda)$ is the continuous part of the Plancherel measure for coclosed $\frac{1}{2}N$ -forms. The constant μ_0 is the contribution from the harmonic forms defined by

$$\mu_0 \equiv \frac{\Omega_{N-1}}{c_{Ng}(\frac{1}{2}N)} \sum \hat{h}^* \cdot \hat{h}(0), \tag{5.8}$$

where the summation is over all normalized harmonic forms [compare this with Eq. (4.8)]. Substituting the solutions given by (5.1) and (5.2) in (5.8) we find

$$\mu_0 = \frac{\pi N}{2^{2N-3}} = 2\pi i \text{res } \mu_{N/2}(\lambda)|_{\lambda=-i/2}. \tag{5.9}$$

6. Group-theoretic derivation of $\mu(\lambda)$

As observed in the introduction, the Plancherel measure for an arbitrary vector bundle E^τ over a noncompact Riemannian symmetric space (SS) G/K can be obtained from the Plancherel measure for scalar functions on the group G . The reason is that the space $L^2(G/K, E^\tau)$ of square-integrable sections of E^τ sits in $L^2(G)$ in a natural way. Indeed it follows easily from the Peter–Weyl theorem for $L^2(K)$ combined with the theorem on inducing a representation “in stages”, that the regular representation π of G on $L^2(G)$ is unitarily equivalent to the direct sum of the induced representations π_τ of G on $L^2(G/K, E^\tau)$ for $\tau \in \hat{K}$ (the set of equivalence classes of unitary irreducible representations of K). Each π_τ appears a number of times equal to $d_\tau \equiv \dim(\tau)$ [27]. Thus we can write

$$\left(\pi, L^2(G)\right) \simeq \sum_{\tau \in \hat{K}} d_\tau \left(\pi_\tau, L^2(G/K, E^\tau)\right), \tag{6.1}$$

where \simeq means equivalence under a unitary operator. The Plancherel theorem for $L^2(G)$ is the direct integral decomposition of π over the set \hat{G} of equivalence classes of irreducible unitary representations (IURs) of G . The measure in this decomposition is known as the Plancherel measure. Let $g \rightarrow U(g)$ be an IUR

of G on a Hilbert space V and let f be a smooth function on G with compact support. Consider the operator

$$U[f] \equiv \int_G f(g)U(g)dg. \quad (6.2)$$

If $\{v_i\}$ is an orthonormal basis of V the formal trace $\sum_i \langle U(g)v_i, v_i \rangle$ is generally ill defined. Define instead

$$\Theta_U[f] \equiv \text{Tr } U[f] \equiv \sum_i \int_G f(g) \langle U(g)v_i, v_i \rangle dg. \quad (6.3)$$

This trace can be shown to exist (and to be independent of the basis) and the map $f \rightarrow \Theta_U[f]$ is called the *global character* of U [25]. This is just a distribution on G , i.e. a continuous linear functional on the space of smooth functions of compact support. Moreover Θ_U is uniquely defined by a locally integrable function $\Theta_U(g)$, in the sense that

$$\Theta_U[f] = \int_G f(g)\Theta_U(g)dg. \quad (6.4)$$

The inversion formula is the expansion of $f(e)$ (e is the identity of the group) in terms of the global characters:

$$f(e) = \int_{\hat{G}} \Theta_U[f] d\mu(U), \quad (6.5)$$

where $d\mu(U)$ is the Plancherel measure. The value of f at any other point (i.e. the analog of Eq. (1.8)) may be obtained by

$$f(g) = \int_{\hat{G}} \text{Tr} [U[f]U(g^{-1})] d\mu(U) = \int_{\hat{G}} \Theta_U[f \circ R_g] d\mu(U). \quad (6.6)$$

These formulae are equivalent to the *Plancherel formula* for $f \in L^1(G) \cap L^2(G)$ [37]:

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} \|U[f]\|^2 d\mu(U), \quad (6.7)$$

where

$$\|U[f]\|^2 = \text{Tr} [U[f]U[f]^\dagger] \quad (6.8)$$

is the Hilbert-Schmidt norm of the operator $U[f]$.

Now let $\hat{G}(\tau)$ be the set of those $U \in \hat{G}$ such that $U|_K \supset \tau$. Let $m(\tau, U)$ denote the multiplicity of τ in $U|_K$ (it is known that $m(\tau, U) \leq d_\tau$, see [15,25]). Combining the Plancherel theorem for $L^2(G)$ with (6.1) gives the

following direct integral decomposition for the induced representation π_τ (see Lemma 1 of Ref. [27]):

$$\pi_\tau \simeq \int_{\hat{G}(\tau)} m(\tau, U) U d\mu(U). \tag{6.9}$$

This result may be regarded as a generalization of the classical Frobenius Reciprocity Theorem to the noncompact case. That is, the multiplicity with which U occurs in π_τ coincides with the multiplicity of τ in $U|_K$. The corresponding direct integral decompositions of $L^2(G)$ and $L^2(G/K, E^\tau)$ may be regarded as the spectral decompositions into eigenspaces of the Casimir operators. In order to determine these decompositions we do not need to know all of the IURs of G . All we need are the IURs with nonzero Plancherel measure, the support of $d\mu(U)$ being in general a proper subset of \hat{G} . It is well known (see e.g. Refs. [25,37]) that for a noncompact semisimple Lie group with finite center the IURs that appear in the Plancherel formula are the so-called generalized principal series (constructed from a complete set of cuspidal parabolic subgroups by the method of induced representation) and the discrete series (the IURs with square-integrable matrix coefficients), which exist if and only if G has a compact Cartan subgroup, i.e., if and only if $\text{rank } G = \text{rank } K$. The Plancherel measure for $L^2(G)$ has been determined explicitly by Harish–Chandra (see Ref. [25]).

We shall now apply these considerations to $G = SO_0(N, 1)$, $K = SO(N)$, $G/K = H^N$. Actually H^N may also be regarded as the coset $\text{Spin}(N, 1)/\text{Spin}(N)$, where $\text{Spin}(N, 1)$ and $\text{Spin}(N)$ denote the double coverings of $SO_0(N, 1)$ and $SO(N)$, respectively. (For $N > 2$ these are also the universal covering groups.) In this way one can discuss spinor bundles over H^N .

It is well known (e.g. [37, vol. II, pp. 41–42]) that $m(\tau, U) = 1$ for any τ and any U . For N odd all Cartan subgroups of G are conjugate and there are no discrete series. For N even there are two conjugacy classes of Cartan subgroups, one for a compact Cartan subgroup contained in K , the other for a noncompact Cartan subgroup with one generator in G/K . In this case there are discrete series. If the irrep τ of $SO(N)$ is contained in some discrete series the vector bundle E^τ has square-integrable eigenmodes (of the Laplacian). An example of this will be seen shortly. For the Lorentz group (more precisely for its double cover $\text{Spin}(N, 1)$) we have the following explicit formulae, due to Hirai [22].

1. $N = 2k + 2$ ($k = 0, 1, 2, \dots$): The principal series representations are denoted by $U_{(i\lambda, \sigma)}$, where $i = \sqrt{-1}$, λ is a real number and $\sigma = (n_1, n_2, \dots, n_k)$ is a row of numbers that are either all integers or all half odd integers satisfying

$$0 \leq n_1 \leq n_2 \leq \dots \leq n_k. \tag{6.10}$$

Notice that σ defines a representation of $M = \text{Spin}(N - 1)$. Define $l_j = n_j + j - \frac{1}{2}$ ($j = 1, 2, \dots, k$), and denote the global character of $U_{(i\lambda, \sigma)}$ by $\Theta_{(\lambda, \sigma)}$. There are two sets of representations in the discrete series denoted by $U_{(n_0, \sigma)}^+$ and $U_{(n_0, \sigma)}^-$, where σ is defined as above and n_0 is integer or half odd integer at the same time as the other labels and satisfies

$$\frac{1}{2} < n_0 \leq n_1 \leq n_2 \leq \dots \leq n_k. \tag{6.11}$$

(For $n_0 = \frac{1}{2}$ we have the two “limits of discrete series” which are IURs but are not square-integrable.) Define l_j ($j = 0, \dots, k$) as above, and denote the sum of the characters of $U_{(n_0, \sigma)}^+$ and $U_{(n_0, \sigma)}^-$ by $\Theta_{(n_0, \sigma)}$. Then the inversion formula (6.5) takes for $k \geq 1$ the form⁴

$$cf(e) = \sum_{0 < l_1 < \dots < l_k} \int_0^\infty iP(-i\lambda, l_1, \dots, l_k) g(\lambda) \Theta_{(\lambda, \sigma)}[f] d\lambda + \sum_{0 < l_0 < l_1 < \dots < l_k} P(l_0, l_1, \dots, l_k) \Theta_{(n_0, \sigma)}[f], \tag{6.12}$$

where $c > 0$ is a normalization constant,

$$g(\lambda) = \begin{cases} \tanh(\pi\lambda), & l_i \text{ half odd integers,} \\ \coth(\pi\lambda), & l_i \text{ integers,} \end{cases} \tag{6.13}$$

and P is the following polynomial, corresponding to the product over the positive roots of the $SO_0(N, 1)$ (or $SO(N + 1)$) Lie algebra:

$$P(x_1, x_2, \dots, x_{k+1}) = x_1 x_2 \dots x_{k+1} \prod_{1 \leq s < r \leq k+1} (x_r^2 - x_s^2). \tag{6.14}$$

For $k = 0$ (i.e., for $SL(2, \mathbb{R})$) the continuous part is the sum of two terms, one with $g(\lambda) = \tanh(\pi\lambda)$ and the other with $g(\lambda) = \coth(\pi\lambda)$, see [25, p. 42]. For a fixed σ the continuous part of the Plancherel measure $d\mu(U_{(i\lambda, \sigma)}) = \mu_\sigma(\lambda) d\lambda$ has the following λ -dependence:

$$\mu_\sigma(\lambda) \propto \lambda(\lambda^2 + l_1^2)(\lambda^2 + l_2^2) \dots (\lambda^2 + l_k^2) g(\lambda), \tag{6.15}$$

where the proportionality constant depends on l_j .

2. $N = 2k + 1$ ($k = 1, 2, \dots$): The principal series representations are labelled by $U_{(i\lambda, \sigma)}$, $\lambda \in \mathbb{R}$, $\sigma = (n_1, n_2, \dots, n_k)$, where the numbers n_i are either all integers or all half odd integers and satisfy

$$|n_1| \leq n_2 \leq \dots \leq n_k. \tag{6.16}$$

⁴ We have corrected the continuous part by a factor of 2. This makes the formula consistent with Ref. [28].

The number n_1 can be negative and again σ defines a representation of $\text{Spin}(N - 1)$. Let $l_j = n_j + j - 1, j = 1, 2, \dots, k$. Then the inversion formula reads

$$cf(e) = \sum_{|l_1| < l_2 < \dots < l_k} \int_0^\infty P(i\lambda, l_1, \dots, l_k) \Theta_{(\lambda, \sigma)}[f] d\lambda, \tag{6.17}$$

where $c > 0$ is a normalization constant and P is a polynomial corresponding to the product over the positive roots of the $SO_0(N, 1)$ (or $SO(N + 1)$) Lie algebra:

$$P(x_1, x_2, \dots, x_{k+1}) = \prod_{1 \leq s < r \leq k+1} (x_r^2 - x_s^2). \tag{6.18}$$

For σ fixed the Plancherel measure is just a polynomial in λ^2 :

$$\mu_\sigma(\lambda) \propto (\lambda^2 + l_1^2)(\lambda^2 + l_2^2) \dots (\lambda^2 + l_k^2). \tag{6.19}$$

Now let τ be an irrep of $\text{Spin}(N)$ and let E^τ be the corresponding homogeneous vector bundle over H^N . As we shall see in Section 9, the coincidence limit of the traced heat kernel for the fields (on a SS G/K) defined by $\tau \in \hat{K}$ (i.e. the sections of E^τ) is given by

$$\text{Tr} K(t) = \int_{\hat{G}(\tau)} m(\tau, U) e^{-\omega_U t} d\mu(U), \tag{6.20}$$

where $-\omega_U$ are the eigenvalues of the second-order Casimir operator ($-A$ in our case), and where the volume of the compact group K is normalized to one. Therefore, in order to find the Plancherel measure for the fields defined by τ we simply have to identify the IURs in the Plancherel formula for $\text{Spin}(N, 1)$ which contain τ upon restriction to $\text{Spin}(N)$. Thus we need the branching rule for $\text{Spin}(N, 1) \supset \text{Spin}(N)$ for principal and discrete series (see e.g. [30]). Again we distinguish the cases with N odd and N even.

Let $N = 2k + 2$ and let τ be the irrep of $\text{Spin}(N)$ labelled by $(f_1, f_2, \dots, f_{k+1})$, where f_i are either all integers or all half odd integers satisfying

$$|f_1| \leq f_2 \leq \dots \leq f_{k+1}, \tag{6.21}$$

and f_1 can be negative. Then the principal series representation $U_{(i\lambda, \sigma)}$ contains τ if and only if

$$|f_1| \leq n_1 \leq f_2 \leq n_2 \leq \dots \leq f_k \leq n_k \leq f_{k+1}. \tag{6.22}$$

Using the branching rule for $\text{Spin}(N) \supset \text{Spin}(N - 1)$ it is easy to see that (6.22) is equivalent to the condition that τ contain the irrep σ of $\text{Spin}(N - 1)$.

Therefore we have the following result: $U_{(i\lambda, \sigma)}$ contains τ if and only if τ contains σ , i.e. in symbols

$$\tau|_{\text{Spin}(N-1)} = \bigoplus_j \sigma_j \supset \sigma \iff U_{(i\lambda, \sigma)}|_{\text{Spin}(N)} \supset \tau. \quad (6.23)$$

This duality is a consequence of the Frobenius Reciprocity Theorem. To see this let us recall that the representations in the principal series of $G = \text{Spin}(N, 1)$ (labelled by (λ, σ) , where λ is a real number and σ is an irrep of $M = \text{Spin}(N-1)$) are induced by the (finite-dimensional) unitary representations of a minimal parabolic subgroup $P = MAN$, where $G = KAN$ is an Iwasawa decomposition of G [35,37]. Since every element of G can be written as kp , with k in K and p in P , induction from P to G looks on K like induction from $P \cap K = M$ to K . It follows that the restriction $U_{(i\lambda, \sigma)}|_K$ is just the representation of $K = \text{Spin}(N)$ unitarily induced by σ , and that for each value of λ the representation space of (λ, σ) may be identified with $L^2(K/M, E^\sigma)$, see Ref. [35, p. 219]. Therefore by the quoted theorem, the multiplicity of a given $\tau \in \hat{K}$ in $U_{(i\lambda, \sigma)}|_K$ coincides with the multiplicity of σ in $\tau|_M$. (This result generalizes to arbitrary Riemannian symmetric spaces G/K if $U_{(i\lambda, \sigma)}$ is a representation in the principal series associated with a minimal parabolic subgroup, see Ref. [37, vol.I, p. 450].)

A more geometric interpretation of Eq. (6.23) is as follows. According to the so-called polar coordinates decomposition [19], a (noncompact) Riemannian SS G/K is diffeomorphic to $A^+ \times K/M$ (up to a zero-measure set), where A^+ is a fundamental domain of the Weyl group in A and M is the centralizer of A in K (i.e. the set of elements in K which commute with A). The coset space K/M is diffeomorphic to the orbits of K in G/K . When a field on G/K (defined by $\tau \in \hat{K}$) is restricted to the K -orbits, we obtain a set of fields on K/M . These fields are sections of vector bundles E^{σ_j} over K/M defined by irreps σ_j of M . In our case the above decomposition reads $H^N \simeq \mathbb{R}^+ \times S^{N-1}$ and the representations σ_j are all different, since for the orthogonal groups a given σ can not appear in τ more than once. For example if τ is the defining vector representation of $SO(N)$, E^τ is the tangent bundle over H^N . When we restrict a vector to S^{N-1} we get a vector and a scalar, i.e. $\tau|_M = \sigma_1 \oplus \sigma_2$, where σ_1 is the trivial representation of M and σ_2 is the vector one.

Now according to (6.23), the principal series that contain τ and enter in the decomposition of the induced representation π_τ (i.e. in the right hand side of (6.9)) are precisely those labelled by (λ, σ_j) . In the previous example we have that the principal series contributing to the harmonic analysis of vector fields over H^N are of the form (λ, σ_1) and (λ, σ_2) . It is possible to show that the matrix coefficients of the irreps (λ, σ_1) give longitudinal vectors ($V_\alpha = \nabla_\alpha f$, with f a scalar function), and the matrix coefficients of the irreps (λ, σ_2) give transverse vectors ($\nabla_\alpha V^\alpha = 0$) (see [2, Prop. 4.1 and 4.2]).

Concerning the discrete series, which exists only for N even, we have the following branching rule. A representation in the discrete series, $U_{(n_0, \sigma)}^\pm$, contains $\tau = (f_1, \dots, f_{k+1})$ if and only if, in addition to (6.22), the following condition is satisfied:

$$\frac{1}{2} < n_0 \leq \pm f_1 \leq n_1. \tag{6.24}$$

Thus f_1 must be nonzero (positive for $U_{(n_0, \sigma)}^+$, and negative for $U_{(n_0, \sigma)}^-$).

Now, let us determine the IURs appearing in the right hand side of (6.9) for p -forms on H^N . For $N = 2k + 2$ the bundle of p -forms (i.e. totally antisymmetric tensors) on H^N is defined by the following irreps τ of $SO(N)$:

$$\begin{aligned} p = 0, 2k + 2: & \quad \tau = (0, \dots, 0); \\ p = 1, 2k + 1: & \quad \tau = (0, \dots, 0, 1); \\ p = 2, 2k: & \quad \tau = (0, \dots, 0, 1, 1); \\ & \quad \vdots \\ p = k, k + 2: & \quad \tau = (0, 1, \dots, 1) \\ p = k + 1: & \quad \tau = (1, \dots, 1) \oplus (-1, 1, \dots, 1) \equiv \tau_+ \oplus \tau_- \end{aligned}$$

The bundles of p -forms and $(N - p)$ -forms correspond to the same τ as a consequence of duality. Notice that for $p = k + 1 = N/2$ the bundle is reducible. From the discussion above we see that the only p -forms contained in the discrete series are for $p = k + 1$, namely $\tau_\pm \subset U_{(n_0, \sigma)}^\pm$, where $(n_0, \sigma) = (1, \dots, 1)$. This identifies the square-integrable k -forms on H^{2k} found in Section 5 (see also Ref. [10]). The discrete part of the Plancherel measure (i.e. $P(l_0, \dots, l_k)$ in Eq. (6.12)) is essentially the formal degree of the discrete series (see Ref. [37, vol.II, p. 407]). In our case $l_j = j + \frac{1}{2}$ and a simple calculation gives

$$P\left(\frac{1}{2}, \frac{3}{2}, \dots, k + \frac{1}{2}\right) = \frac{(2k + 2)!}{(k + 1)!2^{2k+2}} \prod_{s=1}^k (2s)! \tag{6.25}$$

The constant c in (6.12) is determined to be $2\pi^{k+1} \prod_{s=1}^k (2s)!$ using Eq. (6.20) for the scalar case. Then we have

$$\frac{1}{c} P\left(\frac{1}{2}, \frac{3}{2}, \dots, k + \frac{1}{2}\right) = \frac{c_N}{2\Omega_{N-1}} g\left(\frac{1}{2}N\right) \mu_0, \tag{6.26}$$

where μ_0 is given by (5.9). This is half the discrete contribution in (5.7). The reason for this is that the square-integrable harmonic $\frac{1}{2}N$ -forms belong to the reducible representation $U_{(n_0, \sigma)}^+ \oplus U_{(n_0, \sigma)}^-$ of $SO_0(N, 1)$. According to (6.20), each discrete series contributes a term $(1/c)P(1/2, 3/2, \dots, k + 1/2)$ to $TrK(t)$. The result in Section 5 is then reproduced.

Concerning the principal series we obtain [using (6.22)] the following list of irreps $\sigma = (n_i)$ such that $(\lambda, \sigma) \supset \tau$:

$$p = 0, 2k + 2: \quad \sigma = (0, \dots, 0); \quad (6.27)$$

$$p = 1, 2k + 1: \quad \sigma = \begin{cases} (0, \dots, 0), \\ (0, \dots, 0, 1); \end{cases} \quad (6.28)$$

$$p = 2, 2k: \quad \sigma = \begin{cases} (0, \dots, 0, 1), \\ (0, \dots, 0, 1, 1); \end{cases} \quad (6.29)$$

⋮

$$p = k, k + 2: \quad \sigma = \begin{cases} (0, 1, \dots, 1), \\ (1, 1, \dots, 1); \end{cases} \quad (6.30)$$

$$p = k + 1: \quad \sigma = (1, 1, \dots, 1) \text{ for both } \tau_{\pm}. \quad (6.31)$$

For $p = 0, 2k + 2$ and $p = k + 1$ we have a “singlet”, for the other values of p we get a “doublet”. If we form the numbers l_j and calculate the continuous part of the Plancherel measure according to (6.15) we find perfect agreement with the results of Section 4, in the following precise sense. For $p = 1, \dots, k$ the second member of each doublet corresponds (using (6.23)) to coexact p -forms and the Plancherel measure coincides with that obtained in Section 4. The first member of the doublet corresponds to exact p -forms and we verify that the Plancherel measures for exact p -forms and for coexact $(p - 1)$ -forms are equal. For $p = k + 2, \dots, 2k + 1$ the role of the two members of a doublet is reversed, i.e. the first corresponds to coexact forms and the second to exact ones. For $p = 0$ ($p = 2k + 2$) we obtain the right result for coexact (exact) forms, i.e. scalars. Finally for $p = k + 1$ we obtain the right result for exact forms and for coexact ones (with the same Plancherel measure). In all cases we verify the equality of the Plancherel measures for exact p -forms and for coexact $(p - 1)$ -forms and of those for coexact p -forms and $(N - p - 1)$ -forms.

Let $N = 2k + 1$. The branching rule for $(\lambda, \sigma) \supset \tau = (f_1, \dots, f_k)$ is

$$|n_1| \leq f_1 \leq n_2 \leq f_2 \leq \dots \leq n_k \leq f_k. \quad (6.32)$$

This is equivalent to $\tau \supset \sigma$ and (6.23) is again true. The discussion proceeds as before. For $p = 0, 1, \dots, k$ the irreps of $SO(N)$ defining p -forms are given by $f_i = 0$ ($i = 1, \dots, |k - p|$) and $f_i = 1$ ($i = |k - p| + 1, \dots, k$). For $p = k + 1, \dots, 2k + 1$, τ is the same as for $(N - p)$ -forms. All bundles are now irreducible. Applying (6.32) for $p = 0, \dots, k - 1$ (and the corresponding $(N - p)$ -forms) we get the same list of σ 's we had before, i.e. Eqs. (6.27), (6.28), and so on. For $p = k, k + 1$ we obtain a “triplet”, namely $\sigma = (\epsilon, 1, \dots, 1)$ where $\epsilon = 0, \pm 1$.

Using (6.23) we see that for $p = 0$ ($p = 2k + 1$) we obtain the right result for coexact (exact) forms, i.e. scalars. For $p = 1, \dots, k - 1$ the second member of each “doublet” corresponds to coexact p -forms and gives the same Plancherel measure obtained in Section 4 with $c = 2\pi^{k+1}k! \prod_{s=1}^{k-1} (2s)!$ in Eq. (6.17). The first member of each doublet corresponds to exact p -forms. For $p = k + 2, \dots, 2k$ the role of the members of each doublet gets reversed. Finally for $p = k$ the terms of the triplet with $\epsilon = \pm 1$ ($\epsilon = 0$) correspond to coexact (exact) k -forms, while for $p = k + 1$ it is the opposite. A duality operator similar to (5.6) can be defined on coexact k -eigenforms. The two IURs with $\epsilon = \pm 1$ have opposite eigenvalues of this operator. In all cases we again verify the identities (2.2) and (2.3).

7. The ζ -function on H^N and S^N

Let N be odd. The p -form ζ -function (4.7) is well defined for $\text{Re } z > N/2$ and $p \neq \rho$. For $p = \rho$ the spectrum extends to zero, $\omega^{\text{CE}}(\lambda, \rho) = \lambda^2$, and we need to introduce a mass parameter, i.e. to consider the zeta function of $\Delta + m^2$ with $m \neq 0$.

First let $p \neq \rho$. In order to define the ζ -function for the other values of z we perform analytic continuation in z . We define the numbers $\alpha_{k,N}^{(p)}$ by

$$\frac{1}{[\lambda^2 + (\rho - p)^2]} \prod_{j=0}^{\rho} (\lambda^2 + j^2) \equiv \sum_{k=1}^{\rho} \alpha_{k,N}^{(p)} \lambda^{2k}. \tag{7.1}$$

The integration in (4.7) can be performed for $\text{Re } z > N/2$ using Eq. (3.251.2) of Ref. [16]. The result is

$$\begin{aligned} \zeta^{(H)}(z) &= \frac{g(p)}{(4\pi)^{N/2} \Gamma(\frac{1}{2}N)} \\ &\times \sum_{k=1}^{\rho} \alpha_{k,N}^{(p)} |\rho - p|^{2k-2z+1} \frac{\Gamma(k + \frac{1}{2}) \Gamma(z - k - \frac{1}{2})}{\Gamma(z)}. \end{aligned} \tag{7.2}$$

For $p = \rho$ the ζ -function of $\Delta + m^2$ is

$$\zeta^{(H)}(z) = g(\rho) b_N \int_0^{\infty} \frac{\mu(\lambda) d\lambda}{(\lambda^2 + m^2)^z}, \tag{7.3}$$

where

$$b_N = \frac{c_N}{\Omega_{N-1}} = 2^{N-3} \Gamma(\frac{1}{2}N) / \pi^{N/2+1}. \tag{7.4}$$

Defining $\alpha_{k,N}^{(\rho)}$ as in (7.1), with the sum over k in the right hand side starting at zero, we obtain

$$\zeta_{A+m^2}^{(H)}(z) = \frac{g(\rho)}{(4\pi)^{N/2}\Gamma(\frac{1}{2}N)} \times \sum_{k=0}^{\rho} \alpha_{k,N}^{(\rho)} m^{2k-2z+1} \frac{\Gamma(k + \frac{1}{2})\Gamma(z - k - \frac{1}{2})}{\Gamma(z)}. \tag{7.5}$$

The ζ -functions (7.2) and (7.5) exhibit “trivial” zeros at $z = 0, -1, -2, \dots$. They are meromorphic in the complex z -plane with simple poles at

$$z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, -\infty, \tag{7.6}$$

in agreement with general theory [26].

Let now N be even. The ζ -function (4.7) is well defined for $\text{Re } z > N/2$ and for any $p = 0, 1, \dots, N - 1$. Defining the numbers $\beta_{k,N}^{(p)}$ by

$$\frac{1}{[\lambda^2 + (\rho - p)^2]} \prod_{j=\frac{1}{2}}^{\rho} (\lambda^2 + j^2) \equiv \sum_{k=0}^{(N-2)/2} \beta_{k,N}^{(p)} \lambda^{2k}, \tag{7.7}$$

and using the identity

$$\tanh(\pi\lambda) = 1 - \frac{2}{e^{2\pi\lambda} + 1} \tag{7.8}$$

in (4.7) we obtain

$$\zeta^{(H)}(z) = \frac{g(\rho)}{(4\pi)^{N/2}\Gamma(\frac{1}{2}N)} \times \sum_{k=0}^{(N-2)/2} \beta_{k,N}^{(\rho)} \left[|\rho - p|^{2k+2-2z} \frac{\Gamma(k + 1)\Gamma(z - k - 1)}{\Gamma(z)} - 4 \int_0^{\infty} \frac{\lambda^{2k+1} d\lambda}{(e^{2\pi\lambda} + 1)[\lambda^2 + (\rho - p)^2]^z} \right]. \tag{7.9}$$

The last term in this expression is analytic in z . The first term carries only a finite number of simple poles at

$$z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, 1, \tag{7.10}$$

again in agreement with Ref. [26]. (Note that the eigenvalue of the discrete modes for $p = \frac{1}{2}N$ is zero. Hence they are excluded in defining the ζ -function.)

Consider now the zeta function of A acting on coexact p -forms on S^N ,

$$\zeta^{(S)}(z) = \sum_{L=1}^{\infty} D_N(L, p) \omega_L^{-z}, \tag{7.11}$$

where $D_N(L, p)$ are the degeneracies (3.12) and the eigenvalues ω_L , given in (2.7), can be rewritten as

$$\omega_L = (L + \rho)^2 - (\rho - p)^2. \tag{7.12}$$

For $p = 0$ (scalars) Δ has a zero mode which is omitted from $\zeta^{(S)}$.

Now assume that N is odd and $p \neq \rho$. Using Theorem 4.1 and the definition (7.1) we can express the degeneracy $D_N(L, p)$ as an even polynomial in the variable $(L + \rho)$:

$$D_N(L, p) = \mathcal{A}_{N,p} \sum_{k=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} (L + \rho)^{2k}, \tag{7.13}$$

where

$$\mathcal{A}_{N,p} \equiv \frac{\pi D_N(1, p)}{\mu(i(\rho + 1)) [2^{N-2} \Gamma(N/2)]^2} = \frac{2g(p)(-1)^\rho}{(N-1)!}. \tag{7.14}$$

The zeta function (7.11) becomes

$$\begin{aligned} \zeta^{(S)}(z) &= \mathcal{A}_{N,p} \sum_{k=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} \\ &\times \sum_{L=1}^{\infty} \frac{1}{(L + \rho)^{2z-2k}} \left[1 - \left(\frac{\rho - p}{L + \rho} \right)^2 \right]^{-z}. \end{aligned} \tag{7.15}$$

We can now use the binomial expansion

$$(1 - x)^{-z} = \sum_{n=0}^{\infty} \frac{\Gamma(z + n) x^n}{n! \Gamma(z)} \tag{7.16}$$

in (7.15) and exchange the sums over L and n . In fact (7.16) holds for $|x| < 1$ and the condition $(\rho - p)^2 < (L + \rho)^2$ is always satisfied for $p = 0, 1, \dots, N - 1$ and $L = 1, 2, \dots$. We obtain $\zeta^{(S)}(z)$ as a series of Riemann-Hurwitz zeta functions $\zeta_R(z, q)$,

$$\begin{aligned} \zeta^{(S)}(z) &= \mathcal{A}_{N,p} \sum_{k=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} \\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(z + n) (\rho - p)^{2n}}{n! \Gamma(z)} \zeta_R(2z + 2n - 2k, \rho + 1), \end{aligned} \tag{7.17}$$

where [16]

$$\zeta_R(z, q) = \sum_{n=0}^{\infty} (n + q)^{-z}, \quad \text{Re } z > 1. \tag{7.18}$$

The function $\zeta^{(S)}(z)$ is meromorphic in the complex z -plane with simple poles given by (7.6), the same as for $\zeta^{(H)}(z)$. It is not difficult to show that $\zeta^{(S)}(z)$ vanishes at the negative integers but not at $z = 0$. [Recall that $\zeta^{(H)}(0) = 0$.] Indeed using

$$\zeta_R(z, \rho + 1) = \zeta_R(z) - \sum_{l=1}^{\rho} l^{-z}, \tag{7.19}$$

where $\zeta_R(z) = \zeta_R(z, 1)$ is the Riemann zeta function, and $\zeta_R(-2q) = 0$, $q = 1, 2, \dots$ [16], we have from (7.17)

$$\begin{aligned} \zeta^{(S)}(-q) &= -\mathcal{A}_{N,p} \sum_{k=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} \sum_{l=1}^{\rho} l^{2k} \\ &\quad \times \sum_{n=0}^q \frac{q!}{n!(q-n)!} (-1)^n (\rho-p)^{2n} l^{2(q-n)} \\ &= -\mathcal{A}_{N,p} \sum_{k=1}^{\rho} \sum_{l=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} l^{2k} [l^2 - (\rho-p)^2]^q, \end{aligned} \tag{7.20}$$

where we have used the binomial theorem. Remembering (4.18), (7.1) and (7.14) we see that the sum over k can be rewritten as

$$\mathcal{A}_{N,p} \sum_{k=1}^{\rho} (-1)^k \alpha_{k,N}^{(p)} l^{2k} = \frac{D_N(1,p)}{\mu(i(\rho+1))} \mu(il) \propto \prod_{j=0, j \neq |\rho-p|}^{\rho} (j^2 - l^2). \tag{7.21}$$

Now as $l = 1, 2, \dots, \rho$ this vanishes unless $l = |\rho - p|$. Therefore the terms with $l \neq |\rho - p|$ do not contribute to the sum over l in (7.20). Since the contribution of the term with $l = |\rho - p|$ also vanishes, we finally have

$$\zeta^{(S)}(-q) = 0, \quad N \text{ odd}, \quad q = 1, 2, \dots \tag{7.22}$$

For $z = 0$ we obtain instead

$$\zeta^{(S)}(0) = -\frac{D_N(1,p)}{\mu(i(\rho+1))} \mu(i|\rho-p|) = (-1)^{\rho+1}. \tag{7.23}$$

For N odd and $p = \rho$ the degeneracies can still be written as (7.13) with the sum in the right starting at $k = 0$. The zeta function reduces to a finite sum of Riemann-Hurwitz functions

$$\zeta_{p=\rho}^{(S)}(z) = \mathcal{A}_{N,p} \sum_{k=0}^{\rho} (-1)^k \alpha_{k,N}^{(p)} \zeta_R(2z - 2k, \rho + 1). \tag{7.24}$$

Using Eqs. (7.19) and (7.21) (with the sum over k starting at zero) it is easy to see that we can replace $\zeta_R(2z - 2k, \rho + 1)$ by $\zeta_R(2z - 2k)$ in this formula. The function $\zeta_{p=\rho}^{(S)}(z)$ carries only a finite number of poles at

$$z = \frac{1}{2}N, \frac{1}{2}N - 1, \dots, \frac{1}{2}, \tag{7.25}$$

and vanishes at the negative integers, whereas at $z = 0$ it equals

$$\zeta_{\rho=\rho}^{(S)}(0) = -\frac{D_N(1, \rho)}{2\mu(i(\rho + 1))} \mu(0) = (-1)^{\rho+1}. \tag{7.26}$$

Since the residues of the zeta function at $z = \frac{1}{2}N - n$ are proportional to the heat kernel coefficients a_n , $n = 0, 1, 2, \dots$, [4] we see from (7.25) that for N odd and $p = \rho$ the Minakshisundaram-DeWitt expansion of the heat kernel of Δ on S^N terminates at $n = \rho$, i.e. $a_n = 0$ for $n > \rho$.

For N even we can proceed in a similar way. From Theorem 4.1 and (7.7) we can express the degeneracies as an odd polynomial in $(L + \rho)$:

$$D_N(L, p) = B_{N,p} \sum_{k=0}^{(N-2)/2} (-1)^k \beta_{k,N}^{(p)} (L + \rho)^{2k+1}, \tag{7.27}$$

where

$$B_{N,p} \equiv \frac{iD_N(1, p)}{\text{res } \mu|_{\lambda=i(\rho+1)} [2^{N-2}\Gamma(N/2)]^2} = \frac{2g(p)(-1)^{(N-2)/2}}{(N-1)!}. \tag{7.28}$$

The expression for the zeta function, analogous to (7.17), is

$$\begin{aligned} \zeta^{(S)}(z) &= B_{N,p} \sum_{k=0}^{(N-2)/2} (-1)^k \beta_{k,N}^{(p)} \\ &\times \sum_{n=0}^{\infty} \frac{\Gamma(z+n)(\rho-p)^{2n}}{n!\Gamma(z)} \zeta_R(2z+2n-2k-1, \rho+1). \end{aligned} \tag{7.29}$$

The function $\zeta^{(S)}(z)$ is meromorphic in the complex z -plane with simple poles at the same values (7.10) as for $\zeta^{(H)}(z)$.

8. The relation between $\zeta^{(S)}(z)$ and $\zeta^{(H)}(z)$

In the previous section we have determined the analytic continuation of the p -form zeta functions on H^N and S^N . Although they exhibit similar analytic properties, we did not establish a relationship between them. In this section we obtain this relation by means of complex contours.

The eigenvalues ω_L of the Hodge-de Rham operator acting on coexact p -forms on S^N are given by (7.12). The eigenvalues of Δ on H^N may be obtained from ω_L by replacing L by $i\lambda - \rho$ and changing sign, i.e.

$$\omega_\lambda = \lambda^2 + (\rho - p)^2. \tag{8.1}$$

We need to discuss the even and odd dimensional cases separately since the Plancherel measures have different analytic properties in the two cases.

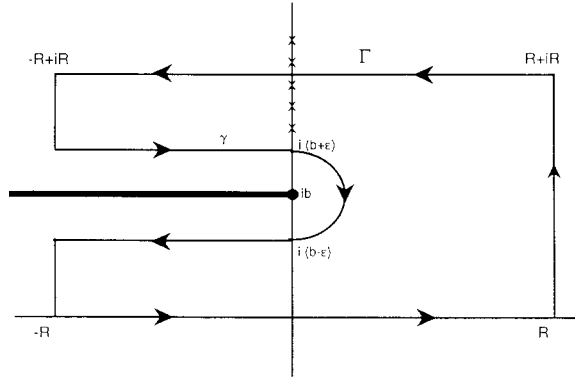


Fig. 1.

8.1. *N* even

Let $b \equiv |\rho - p|$ and consider for $\text{Re } z > \frac{1}{2}N$ the integral

$$I = \oint_{\Gamma} \frac{\mu(\lambda)d\lambda}{(-\lambda^2 - b^2)^z} \tag{8.2}$$

over the contour Γ shown in Fig. 1. We define the phase by letting $(-\lambda^2 - b^2)^z = |-\lambda^2 - b^2|^z$ on the segment $[ib, +i\infty]$. The integrand has two branch points at $\lambda = \pm ib$. The cuts are chosen to run from ib to $ib - \infty$, and from $-ib$ to $-ib + \infty$. The contour Γ is a rectangle of vertices $(-R, R, R + iR, -R + iR)$ deformed around the cut at ib to the contour γ consisting of a small semicircle of radius ϵ centered at ib and of the segments $[i(b + \epsilon) - R, i(b + \epsilon)]$, $[i(b - \epsilon), i(b - \epsilon) - R]$. The point iR lies between consecutive poles, i.e.,

$$\rho + n + 1 < R < \rho + n + 2. \tag{8.3}$$

The integrand in I is analytic inside the contour Γ except for the simple poles on the imaginary axis [see (4.20)]. By applying the residue theorem we obtain

$$I = 2\pi i \sum_{L=1}^{n+1} \text{res} \frac{\mu(\lambda)}{(-\lambda^2 - b^2)^z} \Big|_{\lambda=i(\rho+L)}. \tag{8.4}$$

Consider now the limit $R, n \rightarrow \infty$. As long as $\text{Re } z > \frac{1}{2}N$ the integrals over the sides of Γ other than the real line and γ tend to zero in this limit. To see this define $f(\lambda)$ from Eq. (4.19) according to

$$\mu(\lambda) = \pi \tanh(\pi\lambda)f(\lambda). \tag{8.5}$$

Now the function $\tanh(\pi\lambda)$ is bounded over the sides of Γ and $f(\lambda)$ is a polynomial in λ of order $N - 1$. Therefore the leading terms of the integrals over the sides behave like $1/R^{2\text{Re } z - N}$ and approach zero if $\text{Re } z > \frac{1}{2}N$. Combining this with Theorem 4.1, we obtain as $R \rightarrow \infty$

$$\begin{aligned}
 & 2 \int_0^\infty \frac{\mu(\lambda)d\lambda}{(\lambda^2 + b^2)^z} + \int_\gamma \frac{\mu(\lambda)d\lambda}{(\lambda^2 + b^2)^z} \\
 & = 2\pi i e^{-i\pi z} \frac{\text{res } \mu(\lambda)|_{\lambda=i(1+\rho)}}{D_N(1,p)} \sum_{L=1}^\infty \frac{D_N(L,p)}{[(L+\rho)^2 - b^2]^z}. \tag{8.6}
 \end{aligned}$$

The sum on the right-hand side is just the p -form ζ -function (7.11) on the N -sphere. On the other hand the first integral on the left hand side of (8.6) is proportional to $\zeta^{(H)}(z)$ given in Eq. (4.7). A simple calculation using (4.19) and (3.12) gives

$$\frac{D_N(1,p)}{\text{res } \mu(\lambda)|_{\lambda=i(\rho+1)}} = i\pi(-1)^{N/2}g(p)b_N\Omega_N, \tag{8.7}$$

where $g(p)$ and b_N are given in Eqs. (4.9) and (7.4), and Ω_N is the volume of the N -sphere [see Eq. (3.6)]. Using Eqs. (4.7), (7.11) and (8.7) we can rewrite Eq. (8.6) in the following way:

$$\zeta^{(S)}(z) = e^{i\pi(z-N/2)}\Omega_N \left(\zeta^{(H)}(z) + \frac{1}{2}g(p)b_N \int_\gamma \frac{\mu(\lambda)d\lambda}{(\lambda^2 + b^2)^z} \right). \tag{8.8}$$

To simplify this expression further we use (8.5) and the identity

$$\tanh(\pi\lambda) = -1 + \frac{2}{e^{-2\pi\lambda} + 1} \tag{8.9}$$

in the integral over γ , and show that the term “ -1 ” gives no contribution, i.e.,

$$\int_\gamma \frac{f(\lambda)d\lambda}{(\lambda^2 + b^2)^z} = 0, \quad \text{Re } z > \frac{1}{2}N. \tag{8.10}$$

In order to see this consider for $\text{Re } z > \frac{1}{2}N$ the integral of $f(\lambda)/(\lambda^2 + b^2)^z$ over the contour Γ . Since $f(\lambda)$ is odd [compare Eqs. (8.5) and (4.19)], the integral over the real line vanishes and the integrals over the sides of the rectangle again approach zero as $R \rightarrow \infty$. Since the integrand is analytic and has no poles inside the contour Γ , the integral over the remaining part γ must vanish. This establishes Eq. (8.10).

We thus obtain the following form of $\zeta^{(S)}(z)$, valid for all z :

$$\begin{aligned}
 \zeta^{(S)}(z) & = e^{i\pi(z-N/2)}\Omega_N \\
 & \times \left(\zeta^{(H)}(z) + \pi g(p)b_N \int_\gamma \frac{f(\lambda)d\lambda}{(e^{-2\pi\lambda} + 1)(\lambda^2 + b^2)^z} \right). \tag{8.11}
 \end{aligned}$$

Since the last term in (8.11) is analytic, we see that $\zeta^{(S)}(z)$ carries the same poles and residues as $\zeta^{(H)}(z)$ in Eq. (7.9) [see (7.10)]. The finite parts of these ζ -functions are of course different.

Eq. (8.11) is well suited for analytic continuation to negative values of z . If $\text{Re } z < 0$, it is possible to obtain a simpler expression for the integral over γ . Consider the integral over the semicircle around the point ib . By letting $\lambda - ib = \epsilon e^{i\theta}$ we see that as $\epsilon \rightarrow 0$ this integral behaves as $\epsilon^{-\text{Re } z}$. Thus, if $\text{Re } z < 0$, the contour γ can be contracted to run along the edges of the cut and there is no contribution from the semicircle. By using the phases

$$(\lambda - ib)^z = r^z e^{i\theta z} = \begin{cases} r^z e^{2\pi i z} & \text{above } (\theta = 2\pi), \\ r^z & \text{below } (\theta = 0), \end{cases} \tag{8.12}$$

we obtain

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma} \frac{f(\lambda) d\lambda}{(e^{-2\pi\lambda} + 1)(\lambda^2 + b^2)^z}$$

$$= 2ie^{-i\pi z} \sin(\pi z) \int_{ib}^{ib-\infty} \frac{f(\lambda) d\lambda}{(e^{-2\pi\lambda} + 1)(\lambda^2 + b^2)^z} \tag{8.13}$$

$$= -2ie^{-i\pi z} \sin(\pi z) \int_0^{\infty} \frac{f(ib - y) dy}{[e^{2\pi(y-ib)} + 1] [(y - ib)^2 + b^2]^z}, \tag{8.14}$$

$\text{Re } z < 0$.

Thus the integral along γ in (8.11) vanishes for $z = -1, -2, \dots$. The proportionality of the pole residues of $\zeta^{(S)}(z)$ and $\zeta^{(H)}(z)$, and of their values at the negative integers leads to

$$a_n^{(S^N)} = (-1)^n a_n^{(H^N)} \tag{8.15}$$

(valid for $n = 0, 1, 2, \dots$ and $n \neq \frac{1}{2}N$), where a_n are the (unintegrated) coefficients of the asymptotic (small time) expansion of the trace of the heat kernel of \mathcal{A} acting on coexact forms. [Since S^N and H^N are homogeneous, the coincidence limits $a_n = a_n(x, x)$ are constants independent of x . The integrated coefficients on S^N are simply $u_n = \Omega_N a_n$.] Eq. (8.15) follows from the relations

$$a_n \propto (-1)^{N/2-n} [\Gamma(n + 1 - \frac{1}{2}N)]^{-1} \zeta(\frac{1}{2}N - n),$$

$$n = \frac{1}{2}N, \frac{1}{2}N + 1, \dots, \tag{8.16}$$

$$a_n \propto \Gamma(\frac{1}{2}N - n) \text{res } \zeta(z)|_{z=N/2-n},$$

$$n = 0, 1, \dots, \frac{1}{2}N - 1, \tag{8.17}$$

valid for N even. [The constant of proportionality in (8.16), (8.17) is $(4\pi)^{N/2} / \Omega_N$ for S^N (see e.g. [26]), and $(4\pi)^{N/2}$ for H^N .] However the coefficients

$a_{N/2}$, which by (8.16) are proportional to $\zeta(0)$, are not related as in (8.15).⁵ Indeed for $z = 0$ the contour γ in (8.11) can be deformed to a circumference of radius ϵ going clockwise around the simple pole at $\lambda = ib$. The contribution to $\zeta(0)$ is easily evaluated and we find

$$\zeta^{(S)}(0) = (-1)^{N/2} \Omega_N \zeta^{(H)}(0) + (-1)^{p+1} \quad \text{for } p < \rho. \tag{8.18}$$

If $p > \rho$, then the second term is $(-1)^p$. It is not difficult to verify that this value agrees with that obtained by using Eq. (7.29).

8.2. N odd

For N odd $\mu(\lambda)$ is analytic [see (4.18)]. To use the residue theorem we need to introduce a fictitious spectral function $\tilde{\mu}(\lambda)$ defined by

$$\tilde{\mu}(\lambda) \equiv \coth(\pi\lambda)\mu(\lambda). \tag{8.19}$$

We then define $f(\lambda) \equiv \mu(\lambda)/\pi$. We immediately see that the function $\tilde{\mu}(\lambda)$ has simple poles given again by Eqs. (4.20) and is otherwise analytic in the λ -plane. By calculating the residues at the poles in (4.20) we find

$$\frac{\text{res } \tilde{\mu}(i(L + \rho))}{\text{res } \tilde{\mu}(i(1 + \rho))} = \frac{D_N(L, p)}{D_N(1, p)}, \quad L = 1, 2, \dots \tag{8.20}$$

This is analogous to Theorem 4.1 for N even.

First we treat the case with $p \neq \rho$. Consider the integral of $\tilde{\mu}(\lambda)/(-\lambda^2 - b^2)^z$ (where again $b \equiv |\rho - p|$) over the same contour Γ of Fig. 1 [with R satisfying (8.3)]. Notice that $\tilde{\mu}(\lambda)$ is an odd function (since $\mu(\lambda)$ is even), and therefore the integral over the real line is now zero. By applying the residue theorem we obtain in the limit $R \rightarrow \infty$

$$\zeta^{(S)}(z) = \frac{D_N(1, p)e^{i\pi z}}{\text{res } \tilde{\mu}(\lambda)|_{\lambda=i(1+\rho)}} \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{\mu}(\lambda)d\lambda}{(\lambda^2 + b^2)^z}. \tag{8.21}$$

A simple calculation gives

$$\frac{D_N(1, p)}{\text{res } \tilde{\mu}|_{i(1+\rho)}} = (-1)^p \pi g(p) b_N \Omega_N. \tag{8.22}$$

We now use the identity

$$\coth(\pi\lambda) = -1 - \frac{2}{e^{-2\pi\lambda} - 1} \tag{8.23}$$

in the integral over γ in (8.21). The contribution of the term “ -1 ” can be related to $\zeta^{(H)}(z)$ in the following way. Consider for $\text{Re } z > \frac{1}{2}N$ the integral

⁵This result holds only for $p \neq 0$. For $p = 0$ (i.e. for scalars) it is true that $a_{N/2}^{(S^N)} = (-1)^{N/2} a_{N/2}^{(H^N)}$. However (8.16) fails to hold on S^N for $n = N/2$, as the presence of a zero eigenvalue gives $\zeta^{(S^N)}(0) = (4\pi)^{-N/2} \Omega_N a_{N/2}^{(S^N)} - 1$, see [4]. This agrees with (8.18) for $p = 0$.

over Γ of the function $f(\lambda)/(\lambda^2 + b^2)^z$. Since $f(\lambda)$ is even and the integrand is analytic inside the contour Γ we find

$$\int_{\gamma} \frac{f(\lambda)d\lambda}{(\lambda^2 + b^2)^z} = -2 \int_0^{\infty} \frac{f(\lambda)d\lambda}{(\lambda^2 + b^2)^z}. \tag{8.24}$$

Using Eqs. (8.22), (8.23) and (8.24) in (8.21) we obtain

$$\begin{aligned} \zeta^{(S)}(z) &= -i(-1)^{\rho} e^{i\pi z} \\ &\times \Omega_N \left\{ \zeta^{(H)}(z) + \pi g(p) b_N \int_{\gamma} \frac{f(\lambda)d\lambda}{(1 - e^{-2\pi\lambda})(\lambda^2 + b^2)^z} \right\}. \end{aligned} \tag{8.25}$$

The function $\zeta^{(S)}(z)$ has the same poles and residues as $\zeta^{(H)}(z)$ in Eq. (7.2) [see (7.6)]. Again for $\text{Re } z < 0$ the integral over γ can be contracted to yield a result similar to (8.13) and (8.14). Thus $\zeta^{(S)}(z)$ vanishes at $z = -1, -2, \dots$, in agreement with (7.22). For $z = 0$ the integrand in the second term of (8.25) has a simple pole at $\lambda = ib$ (since $b = |\rho - p|$ is integer). The contour γ can be replaced by a loop around this pole and we obtain

$$\begin{aligned} \zeta^{(S)}(0) &= -i(-1)^{\rho} \Omega_N g(p) b_N (-2\pi i) \operatorname{res} \frac{\mu(\lambda)}{1 - e^{-2\pi\lambda}} \Big|_{\lambda=i|\rho-p|} \\ &= (-1)^{\rho+1} \end{aligned} \tag{8.26}$$

in complete agreement with (7.23).

Finally, we need to consider the case of N odd and $p = \rho$. As already observed in Section 5, the zeta function of the operator Δ on H^N is ill defined in this case because the spectrum extends to zero, and there is no region in the z -plane where the integral in (4.7) converges. We could consider the zeta functions of $\Delta + m^2$ (where $m \neq 0$ is a constant) on S^N and H^N and relate them as before. Instead, let us obtain a contour representation of $\zeta^{(S)}(z)$ of Δ by the method used above, and show its equivalence with (7.24). Consider for $\text{Re } z > \frac{1}{2}N$ the integral

$$I' = \oint_{\Gamma'} \frac{\tilde{\mu}(\lambda)d\lambda}{(-\lambda^2)^z} \tag{8.27}$$

where the λ -plane is cut along the negative real axis (the branch point of the integrand being now at the origin), and the contour Γ' is a square of vertices $(R + iR, -R + iR, -R - iR, R - iR)$ deformed around the cut at $\lambda = 0$ to a contour γ' similar to γ in Fig. 1. The integrand in I' is analytic inside the contour Γ' except for the simple poles on the imaginary axis at $\lambda = \pm i(\rho + L)$, $L = 1, 2, \dots$ (see (4.20)). In the limit $R \rightarrow \infty$ the integrals over the sides of Γ' tend to zero as long as $\text{Re } z > \frac{1}{2}N$. If the correct phases are inserted we find

$$\begin{aligned}
 I' &= 2\pi i \sum \text{residues in upper + lower plane} \\
 &= 2\pi i \frac{\text{res } \tilde{\mu}(\lambda)|_{\lambda=i(1+\rho)}}{D_N(1, \rho)} (1 + e^{2\pi iz}) \zeta_{p=\rho}^{(S)}(z).
 \end{aligned}
 \tag{8.28}$$

Using

$$1 + e^{2\pi iz} = 2e^{i\pi z} \cos(\pi z) \tag{8.29}$$

and Eq. (8.22) (which still holds), we obtain

$$\zeta_{p=\rho}^{(S)}(z) = [4i \cos(\pi z)]^{-1} (-1)^\rho g(\rho) b_N \Omega_N \int_{\gamma'} \lambda^{-2z} \tilde{\mu}(\lambda) d\lambda. \tag{8.30}$$

Using (8.19) and the identity (8.23) in the integral over γ' we see that the term “-1” does not contribute, and we find

$$\zeta_{p=\rho}^{(S)}(z) = \frac{i(-1)^\rho g(\rho) b_N \Omega_N}{2 \cos(\pi z)} \int_{\gamma'} \frac{\mu(\lambda) d\lambda}{(e^{-2\pi\lambda} - 1) \lambda^{2z}}, \tag{8.31}$$

which replaces Eq. (8.25). For $\text{Re } z < 0$ the integral over γ' can be contracted to yield a result similar to (8.13) and (8.14), showing that $\zeta^{(S)}(z)$ vanishes at the negative integers. For $z = 0$ by repeating the calculation that led us to (8.26) we find $\zeta_{p=\rho}^{(S)}(0) = (-1)^\rho$ in agreement with (7.26).

To verify the equivalence of (8.31) and (7.24) directly we use the following contour representation for the Riemann zeta function (see, e.g., Ref. [16, Eq. (9.512)])

$$\int_{\gamma'} \frac{\lambda^{z-1} d\lambda}{e^{-\lambda} - 1} = -2\pi i \frac{\zeta_R(z)}{\Gamma(1-z)}. \tag{8.32}$$

Using this and

$$\mu(\lambda) = \frac{\pi}{[2^{N-2} \Gamma(\frac{1}{2}N)]^2} \sum_{k=0}^{\rho} \alpha_{k,N}^{(\rho)} \lambda^{2k} \tag{8.33}$$

(compare Eqs. (4.18) and (7.1)) in (8.31) we obtain

$$\begin{aligned}
 \zeta_{p=\rho}^{(S)}(z) &= \frac{(-1)^\rho g(\rho) b_N \Omega_N}{2 \cos(\pi z) [2^{N-2} \Gamma(\frac{1}{2}N)]^2} \\
 &\quad \times \sum_{k=0}^{\rho} \alpha_{k,N}^{(\rho)} (2\pi)^{2z-2k} \frac{\zeta_R(2k - 2z + 1)}{\Gamma(2z - 2k)}.
 \end{aligned}
 \tag{8.34}$$

We now use the transformation formula (see, e.g., Ref. [16, Eq. (9.535), n. 2])

$$\frac{\zeta_R(z)}{\Gamma(1-z)} = 2^z \pi^{z-1} \sin\left(\frac{1}{2}\pi z\right) \zeta_R(1-z), \tag{8.35}$$

to find

$$\zeta_{p=\rho}^{(S)}(z) = \frac{(-1)^\rho g(\rho) b_N \Omega_N}{\left[2^{N-2} \Gamma\left(\frac{1}{2} N\right)\right]^2} \sum_{k=0}^{\rho} (-1)^k \alpha_{k,N}^{(\rho)} \zeta_R(2z - 2k). \tag{8.36}$$

This agrees with (7.24) in view of Eqs. (7.14) and (8.22). A similar procedure (though more complicated) should allow one to verify directly the equivalence of (7.29) and (8.11) (for N even), and that of (7.17) and (8.25) (for N odd).

9. Conclusions

The results of this paper and of Ref. [7] (see also [2]) indicate that the eigenmodes of the Laplace–Beltrami operator on vector bundles over the hyperbolic space $H^N = SO_0(N, 1)/SO(N)$ can be determined explicitly by separating variables in geodesic polar coordinates. The relevant equations for the radial modes can be reduced to hypergeometric form. The Plancherel measure can then be obtained in a purely analytic way from the general theory of Sturm–Liouville systems.

This method becomes impracticable for a generic noncompact Riemannian SS G/K , especially for the higher-rank spaces, where the algebra involved is formidable. In these cases it is more useful to develop analysis on SS based on group theoretic methods as we did in Section 6.

Let τ be an irreducible unitary representation of the maximal compact subgroup K on a vector space H_τ . Let $g \rightarrow U(g)$ be an irreducible unitary representation of G in a Hilbert space V such that $U|_K$ contains τ . Define the *generalized spherical functions of type τ* by

$$\Phi_\tau^U(g) = P(\tau)U(g)P(\tau), \tag{9.1}$$

where

$$P(\tau) = d_\tau \int_K U(k^{-1}) \chi_\tau(k) dk \tag{9.2}$$

is the projector of V onto V_τ , the closed subspace of V consisting of those vectors which transform under K according to τ [15]. (Here $d_\tau \equiv \dim H_\tau$, χ_τ is the character of τ , and we have normalized the volume of K to 1.) The functions Φ_τ^U are essentially the matrix coefficients of $U(g)$ in V_τ . Regarded as operators on V_τ , they satisfy

$$\Phi_\tau^U(k g k') = \tau_U(k) \Phi_\tau^U(g) \tau_U(k'), \quad k, k' \in K, \quad g \in G, \tag{9.3}$$

where $\tau_U(k)$ denotes the restriction of $U(k)$ to V_τ . The traces $\phi_\tau^U(g) = \text{Tr}[\Phi_\tau^U(g)]$ are well defined because V_τ is finite-dimensional (this is always

true for a semisimple G).⁶ The functions ϕ_τ^U and their properties were first considered by Godement [15]. It follows from (9.3) that they are K -central, i.e. $\phi_\tau^U(k g k^{-1}) = \phi_\tau^U(g)$. Furthermore from (9.1) and (9.2) we have

$$\phi_\tau^U(g) = d_\tau \int_K \Theta_U(g k^{-1}) \chi_\tau(k) dk, \tag{9.4}$$

where $\Theta_U(g)$ is the character function in Eq. (6.4). Thus the traces of the generalized spherical functions Φ_τ^U are obtained by taking the convolution (in K) of the character of U with the character of τ .

Due to the Cartan decomposition $G = KAK$, the functions Φ_τ^U and their traces are determined by their restriction to the Cartan subspace A . Let \mathbf{X} be a generator of G/K and define $\hat{\Phi}_\tau^U(\text{Exp } \mathbf{X}) = \Phi_\tau^U(\text{exp } \mathbf{X})$ on G/K , where Exp and exp denote the exponential maps on G/K and G , and define $\hat{\phi}_\tau^U$ in a similar way. Using $k \text{Exp } \mathbf{X} = \text{Exp}[\text{Ad}(k)\mathbf{X}]$ and $\text{exp}[\text{Ad}(k)\mathbf{X}] = k \text{exp } \mathbf{X} k^{-1}$ we see that

$$\hat{\phi}_\tau^U(kx) = \hat{\phi}_\tau^U(x), \quad x = \text{Exp } \mathbf{X}, \quad k \in K, \tag{9.5}$$

i.e., the $\hat{\phi}_\tau^U$ are zonal functions on G/K and are thus determined by their restriction to Ax_0 (x_0 is the origin of G/K). The *generalized radial part* of the (second order) Casimir operator of G acting on the restrictions $\Phi_\tau^U(a)$, $a \in A$, is given e.g. in Ref. [37, vol. II, p. 277]. Since the Φ_τ^U are eigenfunctions of the Casimir operator one obtains a differential equation which can in principle be solved explicitly (at least in the rank-one case and for some simple τ).

It can be shown (using (6.9)) that the trace of the heat kernel for the space $L^2(G/K, E^\tau)$ can be expanded in terms of the functions $\hat{\phi}_\tau^U$, where U runs over the irreducible unitary representations of G which contain τ and have nonzero Plancherel measure. Thus we have the following formula:

$$\text{Tr } K(x_0, x, t) = \frac{1}{d_\tau} \int_{\hat{G}(\tau)} \hat{\phi}_\tau^U(x) e^{-t\omega_U} d\mu(U), \tag{9.6}$$

where $-\omega_U$ are the eigenvalues of the second-order Casimir operator acting on Φ_τ^U . In general the right hand side of (9.6) will contain both an integral (corresponding to the principal series) and a sum (corresponding to the discrete series). Eq. (6.20) is the coincidence limit $x \rightarrow x_0$ of (9.6). The “complementary series” of G do not contribute to the harmonic analysis of vector bundles over G/K .⁷ Since the Plancherel formula for $L^2(G)$ is known explicitly by the work of Harish–Chandra [25,37], we see that the spectral decomposition

⁶ It is known that the multiplicity ξ_U of τ in $U|_K$ satisfies $\xi_U \leq d_\tau$ [15,25]. Thus $\dim V_\tau = \xi_U d_\tau$ and the representation τ_U of K on V_τ is the direct sum of ξ_U copies of τ .

⁷ Notice that this is not true for a (pseudo-Riemannian) symmetric space G/H , where H is a noncompact subgroup of G . For example for $N > 2$ the complementary series of $G = SO_0(N+1, 1)$ appear already in the scalar bundle $L^2(G/H)$ where $H = SO_0(N, 1)$ [29].

of $L^2(G/K, E^\tau)$ into eigenspaces of the Laplacian is known, in principle, for any noncompact Riemannian SS G/K . By analytic continuation one can also handle vector bundles over the compact “dual” space U/K (see Ref. [18] for the definition and classification of dual symmetric spaces).

It is then natural to conjecture that Theorem 4.1 can be generalized to any field defined on a pair of dual Riemannian symmetric spaces $G/K - U/K$. That is, the pole residues of the continuous part of the Plancherel measure for the principal series of G containing τ are proportional to the dimensions of the irreps of U which contain τ . In the case of scalars this result was stated, without proof, in Refs. [33,20,4]. The general proof for scalars, based on the work of Vretare [32], has been obtained by Helgason [21]. It is evident that a general proof for arbitrary vector bundles on $G/K - U/K$ can be obtained by using the Harish–Chandra formula for the Plancherel measure on \hat{G} (this formula will reduce to the Weyl dimension formula when calculating the residues).

Concerning the explicit form of the eigenmodes of the Laplacian and of the spherical functions Φ_τ^U not much is known in general. [The Plancherel measure is known because it is determined only by the asymptotic form of the spherical functions at infinity.]

For example in the scalar case the eigenfunctions of the Laplacian on G/K (spherical or not) are known only in very few cases, e.g., in the rank-one case and in the complex case. Formally the scalar spherical functions on G/K can be obtained by “averaging” the complex characters of A over the compact subgroup K (see [19, p. 418]).

A similar formula for the functions ϕ_τ^U , τ arbitrary, may be obtained when U is in the principal series associated with a minimal parabolic subgroup (see Ref. [37, vol. II, pp. 41–42]). However, except for some very simple cases (e.g. $G = SL(2, R)$), closed expressions do not seem to be known. It would be interesting if one could solve explicitly the differential equation for Φ_τ^U (or their traces) in the rank-one case and for some simple choice of τ .

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